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Perturbed integral equation method on determining unknown geometry in fluid flow

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Abstract

We consider an inverse problem arising in fluid flow. An algorithm to find the shape of a body in uniform flow is proposed when the tangential velocity on its boundary is given a priori. The fluid flow is assumed to be inviscid, incompressible and irrotational.

The essential idea to develop our algorithm is the boundary modification process toward the solution shape with the help of the perturbed integral equations. The perturbed integral equations are derived from the boundary perturbation. We also give examples exhibiting the reliability for our proposed algorithm. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The applicable inverse problems are dealt with by many researchers in the field of engineering and applied mathematics [1,3,4,8]. Especially, many inverse problems determining the unknown geometry are closely related with the design problems to improve the efficiency of mechanism.

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In this paper, we determine the two-dimensional body profile in uniform potential flow on which the tangential speed is given a priori. As a typical example of such a problem, we can take a wing design in aerodynamics. As a matter of fact, in potential flow, the tangential speed of a fluid on the boundary of a body is closely related with the pressure on it, which is called the Bernoulli theory. Hence, if we can determine the body shape under consideration, we can design the shape of the body with the desired pressure distribution on its boundary.

However, the inverse problem determining the shape is highly nonlinear, so it is not easy to develop a robust and efficient method without trial and error. Hence, an intrinsic observation of this problem is needed to overcome the difficulties. We suggest an algorithm providing substantial improvement, based on careful insights into the problem. In the present paper, we assume mainly the symmetry of a body with respect to the axis coinciding with the outer flow direction. We are mainly interested in the development of the algorithm rather than the use of our technique in a practical situation. Nevertheless, we also provide a nonsymmetric example.

In a similar situation to our problem for fluid flow, Zedan and Dalton [10] used an axial source distribution to find the shape, Dinavahi and Chow [4] adopted vortex rings. They consider only the symmetric cases. Our method also works on three-dimensional axisymmetric cases.

In Section 2, we state our inverse problem to find a shape satisfying a given tangential velocity. We also define a direct problem associated with the inverse problem. In Section 3, to find the solution shape proposed in Section 2, we derive perturbed boundary integral equations which will be used to develop our algorithm. We also give an example showing the reliability of these equations. In Section 4, we propose our main algorithm to find the solution shape. In Section 5, several examples are given. In Section 5.1, there are several symmetric examples, which have steep shape, smooth shape, front and rear angled shape with nonzero curvature, and so on. In Section 5.2, a nonsymmetric example is given.

2. Problem statement

In this section, we introduce a direct problem first, and then describe an inverse problem, which is our problem.

First, let us define the direct problem for the fluid flow problem concerned with our inverse problem defined in this paper.

The flow considered in this paper is inviscid, incompressible and irrotational with uniform flow $\mathbf{U} = (U, 0)$ at the far field and it passes around a body. Let Ω be a multiply connected domain in $\mathbb{R}^2$ bounded internally by the surface of a body Γ at rest. We assume the uniform flow $\mathbf{U} = (U, 0)$ is specified at the far field. In order to find the velocity field $\mathbf{u} = (u, v)$ for the two-dimensional incompressible flow, it is convenient to introduce the stream function $\Psi(\mathbf{z})$ such that

$$\frac{\partial \Psi}{\partial y} = u, \quad \frac{\partial \Psi}{\partial x} = -v, \tag{2.1}$$

where $\mathbf{z} = (x, y)$ is a point in the flow domain in $\mathbb{R}^2$. When vortices with strength $\mu(\mathbf{w})$ at $\mathbf{w} \in \Gamma$ are properly distributed on Γ, the induced flow field $\mathbf{u}$ from the vortices on Γ has the circulation Λ

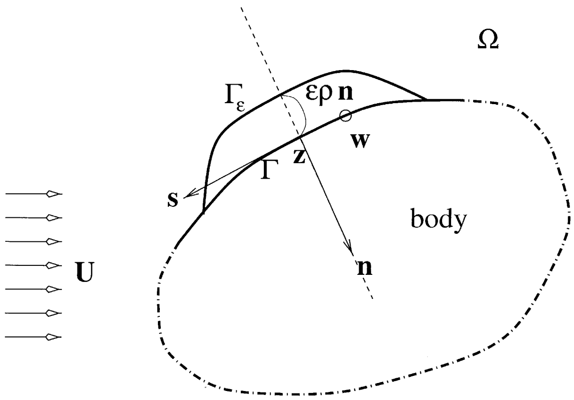

Fig. 1. $(\mathbf{n}, \mathbf{s})$ coordinate system on a curve and perturbation.

which is defined by

$$\Lambda = \int_{\Gamma} \mathbf{u}(\mathbf{z}) \cdot \mathbf{s}(\mathbf{z})\, \mathrm{d}\Gamma_{\mathbf{z}}, \tag{2.2}$$

where $\mathbf{s}(\mathbf{z})$ is the unit tangential vector at $\mathbf{z} \in \Gamma$ as shown in Fig. 1. Furthermore, the stream function Ψ describing the flow can be written as follows:

$$\Psi(\mathbf{z}) = Uy + \int_{\Gamma} \mu(\mathbf{w}) \log|\mathbf{z} - \mathbf{w}|\, \mathrm{d}\Gamma_{\mathbf{w}} + C, \quad \mathbf{z} = (x, y) \in \Omega, \tag{2.3}$$

where μ is a continuous function on Γ and C is a constant. Here, μ and C are to be determined in order that the value of Ψ on Γ is zero and that (2.2) is satisfied for given Λ. Although the stream function Ψ in (2.3) is defined in the exterior of Γ ($\mathbf{z} \in \Omega$), we may consider Ψ as a function defined inside of Γ since the function on the right-hand side of (2.3) is well defined in $\bar{\Omega}^c$. The former and the latter are denoted by Ψ^- and Ψ^+, respectively, which are harmonic with zero boundary values.

Let $\mathbf{n}$ be the unit outward normal vector to Γ. Since the surface of a body itself is a stream line, the velocity on either side of Γ can be written as

$$\mathbf{u}^{\pm}|_{\Gamma} = q^{\pm}\mathbf{s} = \frac{\partial \Psi^{\pm}}{\partial \mathbf{n}}\, \mathbf{s}, \tag{2.4}$$

where q^- and q^+ are the fluid speeds on the outside and on the inside of Γ, respectively.

For any $\mathbf{z} \in \Gamma$, the following are obtained by differentiating $\Psi^{\pm}$ along the vector $\mathbf{n}$ (see [9]):

$$\frac{\partial \Psi^+}{\partial n}(\mathbf{z}) = U\mathbf{e}_2 \cdot \mathbf{n}(\mathbf{z}) + \pi\mu(\mathbf{z}) + \int_{\Gamma} \mu(\mathbf{w}) \frac{\mathbf{n}(\mathbf{z}) \cdot (\mathbf{z} - \mathbf{w})}{|\mathbf{z} - \mathbf{w}|^2}\, \mathrm{d}\Gamma_{\mathbf{w}}, \tag{2.5}$$

$$\frac{\partial \Psi^-}{\partial n}(\mathbf{z}) = U\mathbf{e}_2 \cdot \mathbf{n}(\mathbf{z}) - \pi\mu(\mathbf{z}) + \int_{\Gamma} \mu(\mathbf{w}) \frac{\mathbf{n}(\mathbf{z}) \cdot (\mathbf{z} - \mathbf{w})}{|\mathbf{z} - \mathbf{w}|^2}\, \mathrm{d}\Gamma_{\mathbf{w}}. \tag{2.6}$$

Subtracting (2.6) from (2.5), we have the following equations:

$$2\pi\mu(\mathbf{z}) = \frac{\partial \Psi^+}{\partial n}(\mathbf{z}) - \frac{\partial \Psi^-}{\partial n}(\mathbf{z}) = q^+(\mathbf{z}) - q^-(\mathbf{z}). \tag{2.7}$$

Since the surface of the body is a stream line and we assume that the stream function Ψ is zero on Γ, we have

$$Uy + \int_\Gamma \mu(\mathbf{w}) \log |\mathbf{z} - \mathbf{w}| \, \mathrm{d}\Gamma_{\mathbf{w}} + C = 0 \quad \forall \mathbf{z} \in \Gamma. \tag{2.8}$$

Since Ψ^+ is a bounded harmonic function and the continuity of Ψ up to boundary implies that $\Psi^+(\mathbf{z}) = \Psi^-(\mathbf{z}) = 0$ for all $\mathbf{z} \in \Gamma$, from the maximum principle we have $\Psi^+ \equiv 0$. Thus, we have $q^+ = 0$ on Γ. Therefore, we have the following relationship between μ and q^- such that

$$\mu(\mathbf{z}) = -\frac{1}{2\pi} q^-(\mathbf{z}). \tag{2.9}$$

Throughout the paper, q^- will be denoted by q, which represents the speed of fluid on Γ.

Therefore, let us summarize the above statements: If the circulation Λ and the shape of a body Γ are given, then there exist unique μ and C which satisfy the system of equations,

$$\Lambda = -2\pi \int_\Gamma \mu(\mathbf{z}) \, \mathrm{d}\Gamma_{\mathbf{z}}, \tag{2.10}$$

$$Uy + \int_\Gamma \mu(\mathbf{w}) \log |\mathbf{z} - \mathbf{w}| \, \mathrm{d}\Gamma_{\mathbf{w}} + C = 0 \quad \forall \mathbf{z} \in \Gamma. \tag{2.11}$$

The problem to find μ and C is called the direct problem. For the unique solvability of the direct problem, refer to [7]. The solution μ and C of (2.10) and (2.11) are closely associated with the circulation Λ, which is explained in detail in Appendix A.

Secondly, we introduce the inverse problem corresponding to the direct problem (2.10) and (2.11). The direct problem is to find the solution pair (μ, C) of (2.10) and (2.11) for a given data (Γ, Λ). The inverse problem is to find Γ when μ (or q) is given and Λ is fixed. We define our inverse problem in the following:

Inverse problem (IP). *When the flow speed is given with the distribution q as a function of x, we find the body surface Γ as a function of x whose relationship with q is given by* (2.10) *and* (2.11).

This kind of problem has the ambiguity such that we assign the tangential speed $q(\mathbf{z})$ to point $\mathbf{z} \in \Gamma$ a priori even though Γ is unknown. To circumvent this, we assume the shape of a body is symmetric with respect to the x-axis parallel to the direction of far field uniform flow. The upper half part of the shape of the body, denoted by Γ^+, is assumed to be the graph of a continuous function of x, i.e.,

$$\Gamma^+ = \{(x, f(x)) \in \mathbb{R}^2 \mid f : [-a, a] \to \mathbb{R},\ x \in [-a, a],\ f(-a) = f(a) = 0\}. \tag{2.12}$$

Thus, the velocities at $(\pm a, 0)$ are zero and the flow is symmetric.

However, for nonsymmetric case, we consider the shape is parametrized by the angle and we can define the tangential velocity in advance.

3. Mathematical formulation

For the problem to find the shape of a body under some constraints, the essential step is the shape modification from the previous configuration of the surface of a body satisfying the constraints. In this section, we will derive the perturbed integral equations from the direct problem, which are used to construct our numerical algorithm based on the modification of the shape of a body to find the solution shape. We will derive the perturbed integral equation using the usual $(\mathbf{n}, \mathbf{s})$-coordinate system on the boundary of a body where $\mathbf{n}$ is the outward normal vectors, and $\mathbf{s}$ is the unit tangential vector to the surface of a body Γ.

We let Γ be the smooth boundary of a body, and Γ_ε the perturbed boundary from Γ obtained by a small perturbation $\mathbf{w}_\varepsilon$ defined by

$$\mathbf{w}_\varepsilon = \mathbf{w} + \varepsilon\rho(\mathbf{w})\mathbf{n}(\mathbf{w}), \quad \mathbf{w} \in \Gamma, \tag{3.1}$$

where $\rho(\mathbf{w})$ is a sufficiently smooth function defined on Γ and $\varepsilon > 0$ is a small real number (see Fig. 1).

Let Ω_ε denote the perturbed domain of Ω obtained by the boundary perturbation (3.1), and Γ_ε denote its boundary. If we denote by Ψ_ε the stream function describing the flow around Γ_ε such that

$$\Psi_\varepsilon(\mathbf{z}) = Uy_\varepsilon + \int_{\Gamma_\varepsilon} \mu_\varepsilon(\mathbf{w}_\varepsilon) \log|\mathbf{z} - \mathbf{w}_\varepsilon| \, \mathrm{d}\Gamma_{\mathbf{w}_\varepsilon} + C_\varepsilon, \quad \mathbf{z} \in \Omega_\varepsilon, \tag{3.2}$$

then we have two sets of integral equations on Γ as follows:

Direct problem on Γ

$$\Lambda = -2\pi \int_\Gamma \mu(\mathbf{w}) \, \mathrm{d}\Gamma_{\mathbf{w}}, \tag{3.3}$$

$$-Uy - C = \int_\Gamma \mu(\mathbf{w}) \log|\mathbf{z} - \mathbf{w}| \, \mathrm{d}\Gamma_{\mathbf{w}} \quad \forall \mathbf{z} \in \Gamma. \tag{3.4}$$

Perturbed problem on Γ

$$\Lambda = -2\pi \int_{\Gamma_\varepsilon} \mu_\varepsilon(\mathbf{w}_\varepsilon) \, \mathrm{d}\Gamma_{\mathbf{w}_\varepsilon}, \tag{3.5}$$

$$-Uy_\varepsilon - C_\varepsilon = \int_{\Gamma_\varepsilon} \mu_\varepsilon(\mathbf{w}_\varepsilon) \log|\mathbf{z}_\varepsilon - \mathbf{w}_\varepsilon| \, \mathrm{d}\Gamma_{\mathbf{w}_\varepsilon} \quad \forall \mathbf{z}_\varepsilon \in \Gamma_\varepsilon, \tag{3.6}$$

where $\mathbf{z}_\varepsilon = \mathbf{z} + \varepsilon\rho(\mathbf{z})\mathbf{n}(\mathbf{z})$ and y and y_ε are the y-coordinates of $\mathbf{z}$ and $\mathbf{z}_\varepsilon$, respectively.

We expand the perturbed strength $\mu_\varepsilon(\mathbf{w}_\varepsilon)$ on Γ_ε in (3.6) in terms of ε,

$$\mu_\varepsilon(\mathbf{w}_\varepsilon) = \mu(\mathbf{w}) + \varepsilon\mu_1(\mathbf{w}) + \mathrm{O}(\varepsilon^2), \tag{3.7}$$

for any $\mathbf{w} \in \Gamma$, where $\mathbf{w}_\varepsilon = \mathbf{w} + \varepsilon\rho(\mathbf{w})\mathbf{n}(\mathbf{w})$. We also expand $C_\varepsilon = C + \varepsilon C_1$.

Let s be the arc-length parameter of the curve Γ. Then we have the arc-length variation of Γ_ε for the small boundary perturbation (3.1),

$$|\Gamma'_\varepsilon(s)| = 1 - \varepsilon\kappa(s)\rho(s) + \mathrm{O}(\varepsilon^2), \tag{3.8}$$

where $\kappa(s)$ is the curvature of Γ at s. From the definition of $\mathbf{z}_\varepsilon$ and $\mathbf{w}_\varepsilon$, we have

$$\mathbf{z}_\varepsilon - \mathbf{w}_\varepsilon = (\mathbf{z} - \mathbf{w}) + \varepsilon(\rho(\mathbf{z})\mathbf{n}(\mathbf{z}) - \rho(\mathbf{w})\mathbf{n}(\mathbf{w})) \tag{3.9}$$

and we can calculate the perturbed kernel function in (3.6) as follows:

$$\log|\mathbf{z}_\varepsilon - \mathbf{w}_\varepsilon| = \log|\mathbf{z} - \mathbf{w}| + \varepsilon\left(\rho(\mathbf{z})\frac{\partial}{\partial n_{\mathbf{z}}}\log|\mathbf{z} - \mathbf{w}| + \rho(\mathbf{w})\frac{\partial}{\partial n_{\mathbf{w}}}\log|\mathbf{z} - \mathbf{w}|\right) + \mathrm{O}(\varepsilon^2). \tag{3.10}$$

Substituting the perturbations (3.7), (3.10) and (3.8) into the integral equation (3.6) and collecting terms in each ε order, we can obtain the perturbed boundary integral identity:

$$\begin{aligned}\int_{\Gamma_\varepsilon} \mu_\varepsilon(\mathbf{w}_\varepsilon)\log|\mathbf{z}_\varepsilon - \mathbf{w}_\varepsilon|\,\mathrm{d}\Gamma_{\mathbf{w}_\varepsilon} &= \int_\Gamma \mu(\mathbf{w})\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} \\ &\quad + \varepsilon\left[-\int_\Gamma \kappa(\mathbf{w})\rho(\mathbf{w})\mu(\mathbf{w})\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}}\right. \\ &\quad + \rho(\mathbf{z})\int_\Gamma \mu(\mathbf{w})\frac{\partial}{\partial n_{\mathbf{z}}}\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} \\ &\quad + \int_\Gamma \rho(\mathbf{w})\mu(\mathbf{w})\frac{\partial}{\partial n_{\mathbf{w}}}\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} \\ &\quad \left. + \int_\Gamma \mu_1(\mathbf{w})\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}}\right] + \mathrm{O}(\varepsilon^2). \end{aligned} \tag{3.11}$$

Here, using the following approximation,

$$-Uy_\varepsilon = -U[y(\mathbf{z}) + \varepsilon\rho(\mathbf{z})\mathbf{e}_2 \cdot \mathbf{n}(\mathbf{z})] + \mathrm{O}(\varepsilon^2), \tag{3.12}$$

incorporated with the equations (3.6), (3.4) and (3.11), we can obtain the equations between μ_1 and ρ such that

Perturbed boundary integral equations (*PBIE*)

$$-Uy - C = \int_\Gamma \mu(\mathbf{w})\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}}, \tag{3.13}$$

$$\begin{aligned}-U\rho(\mathbf{z})\mathbf{e}_2 \cdot \mathbf{n}(\mathbf{z}) - C_1 &= -\int_\Gamma \kappa(\mathbf{w})\rho(\mathbf{w})\mu(\mathbf{w})\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} \\ &\quad + \rho(\mathbf{z})\int_\Gamma \mu(\mathbf{w})\frac{\partial}{\partial n_{\mathbf{z}}}\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} \\ &\quad + \int_\Gamma \rho(\mathbf{w})\mu(\mathbf{w})\frac{\partial}{\partial n_{\mathbf{w}}}\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} \\ &\quad + \int_\Gamma \mu_1(\mathbf{w})\log|\mathbf{z} - \mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} \quad \forall \mathbf{z} \in \Gamma, \end{aligned} \tag{3.14}$$

where κ is the curvature with respect to $(\mathbf{n}, \mathbf{s})$ coordinate system on Γ.

Here, if we use Eq. (2.6) and the following fact:

$$\frac{\partial \Psi^-}{\partial n}(\mathbf{z}) = -2\pi\mu(\mathbf{z}),$$

the perturbed integral equation (3.14) can be rewritten as follows:

$$C_1 - \pi\rho(\mathbf{z})\mu(\mathbf{z}) - \int_\Gamma \kappa(\mathbf{w})\rho(\mathbf{w})\mu(\mathbf{w}) \log|\mathbf{z}-\mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} + \int_\Gamma \rho(\mathbf{w})\mu(\mathbf{w})\frac{\partial}{\partial n_{\mathbf{w}}} \log|\mathbf{z}-\mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}}$$
$$= -\int_\Gamma \mu_1(\mathbf{w}) \log|\mathbf{z}-\mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}}. \tag{3.15}$$

To determine the constant C_1 in (3.14), we need one more equation. From a similar procedure used in deriving (3.14), we can obtain the constant circulation condition using Eqs. (3.3) and (3.5), so that we have

Constant circulation condition (*CCC*)

$$\int_\Gamma \kappa(\mathbf{w})\rho(\mathbf{w})\mu(\mathbf{w})\,\mathrm{d}\Gamma_{\mathbf{w}} = \int_\Gamma \mu_1(\mathbf{w})\,\mathrm{d}\Gamma_{\mathbf{w}}. \tag{3.16}$$

For simplicity, we define the following singular integral operators:

$$I_\Gamma[f](\mathbf{z}) = f(\mathbf{z}), \tag{3.17}$$

$$K_\Gamma[f](\mathbf{z}) = \int_\Gamma \kappa(\mathbf{w}) f(\mathbf{w}) \log|\mathbf{z}-\mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}}, \tag{3.18}$$

$$F_\Gamma[f](\mathbf{z}) = \int_\Gamma f(\mathbf{w}) \log|\mathbf{z}-\mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}}, \tag{3.19}$$

$$D_\Gamma[f](\mathbf{z}) = \int_\Gamma f(\mathbf{w}) \frac{(\mathbf{z}-\mathbf{w})\cdot\mathbf{n}(\mathbf{z})}{|\mathbf{z}-\mathbf{w}|^2}\,\mathrm{d}\Gamma_{\mathbf{w}}, \tag{3.20}$$

$$D^*_\Gamma[f](\mathbf{z}) = \int_\Gamma f(\mathbf{w}) \frac{(\mathbf{z}-\mathbf{w})\cdot\mathbf{n}(\mathbf{w})}{|\mathbf{z}-\mathbf{w}|^2}\,\mathrm{d}\Gamma_{\mathbf{w}}. \tag{3.21}$$

Then the integral equation (3.15) can be rewritten as the integral operator form,

$$C_1 - \pi I_\Gamma[\rho\mu](\mathbf{z}) - K_\Gamma[\rho\mu](\mathbf{z}) - D^*_\Gamma[\rho\mu](\mathbf{z}) = -F_\Gamma[\mu_1](\mathbf{z}). \tag{3.22}$$

We have derived the perturbed integral equations (3.16) and (3.15) formally. The example in Appendix B convinces us the reliability of these equations.

Here, it should be pointed out that the solution ρ of the perturbed boundary integral equations (3.16) and (3.15) may not be unique for given shape Γ and μ_1. We provide an example showing the nonuniqueness of ρ in Appendix C.

The problem in the case of symmetric body is treated here. Assume that the body profile Γ of the body is symmetric with respect to the x-axis which is parallel to the outer flow direction. Hence the flow field must be symmetric. We also assume $\Lambda = 0$.

Let us decompose the profile Γ into Γ^+ and Γ^- as shown in Fig. 2. The quantities μ, μ_1, ρ and κ appearing in an arbitrarily shaped body have the symmetries of

$$\kappa(\bar{\mathbf{z}}) = \kappa(\mathbf{z}), \quad \rho(\bar{\mathbf{z}}) = \rho(\mathbf{z}), \tag{3.23}$$

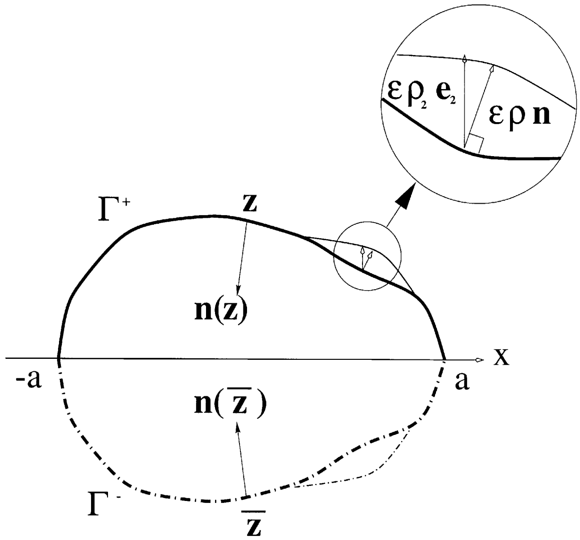

Fig. 2. Symmetric case.

$$\mu(\bar{\mathbf{z}}) = -\mu(\mathbf{z}) \quad \text{and} \quad \mathbf{n}(\bar{\mathbf{z}}) = \overline{\mathbf{n}(\mathbf{z})}, \tag{3.24}$$

where the over-bar means reflection about the symmetric axis. From these symmetries we have the following formulae:

$$K_{\Gamma^+}[\mu\rho](\mathbf{z}) = \int_{\Gamma^+} \kappa(\mathbf{w})\rho(\mathbf{w})\mu(\mathbf{w}) \log \frac{|\mathbf{z}-\mathbf{w}|}{|\mathbf{z}-\bar{\mathbf{w}}|}\, \mathrm{d}\Gamma_{\mathbf{w}}, \tag{3.25}$$

$$F_{\Gamma^+}[\mu_1](\mathbf{z}) = \int_{\Gamma^+} \mu_1(\mathbf{w}) \log \frac{|\mathbf{z}-\mathbf{w}|}{|\mathbf{z}-\bar{\mathbf{w}}|}\, \mathrm{d}\Gamma_{\mathbf{w}}, \tag{3.26}$$

$$D_{\Gamma^+}[\mu](\mathbf{z}) = \int_{\Gamma^+} \mu(\mathbf{w}) \left[\frac{(\mathbf{z}-\mathbf{w})\cdot\mathbf{n}(\mathbf{z})}{|\mathbf{z}-\mathbf{w}|^2} - \frac{(\mathbf{z}-\bar{\mathbf{w}})\cdot\mathbf{n}(\mathbf{z})}{|\mathbf{z}-\bar{\mathbf{w}}|^2} \right] \mathrm{d}\Gamma_{\mathbf{w}}, \tag{3.27}$$

$$D^*_{\Gamma^+}[\mu\rho](\mathbf{z}) = \int_{\Gamma^+} \rho(\mathbf{w})\mu(\mathbf{w}) \left[\frac{(\mathbf{z}-\mathbf{w})\cdot\mathbf{n}(\mathbf{w})}{|\mathbf{z}-\mathbf{w}|^2} - \frac{(\mathbf{z}-\bar{\mathbf{w}})\cdot\overline{\mathbf{n}(\mathbf{w})}}{|\mathbf{z}-\bar{\mathbf{w}}|^2} \right] \mathrm{d}\Gamma_{\mathbf{w}}. \tag{3.28}$$

From our symmetry assumption, we can fix the front-end and the rear-end points of a body at which the boundary of the body intersects with the x-axis of the symmetry line, for example, locating the front-end and the rear-end point at $(-a,0)$ and $(a,0)$, respectively. Furthermore the body shape is assumed to be a function of x variable, i.e., if Γ^+ is the upper boundary of a body, then it can be parameterized by

$$\Gamma^+\colon\ y = f(x), \quad -a \leqslant x \leqslant a.$$

If we consider the boundary perturbation in this situation as follows:

$$\mathbf{w}_\varepsilon(x) = \mathbf{w}(x) + \varepsilon\rho_2(x)\mathbf{e}_2, \tag{3.29}$$

then the following relationship between ρ at (3.1) and ρ_2 is obtained

$$\rho_2(x) = \rho(x)\sqrt{1 + (f'(x))^2}, \tag{3.30}$$

where $\mathbf{e}_2 = (0, 1)$. Thus, the equivalent boundary variation (3.30) is applicable in the symmetric case from previously calculated values of ρ. The symmetric flow assumption makes all of constants C, C_ε, and C_1 zero. In fact, under the assumption of the symmetry, we have the two equations

$$Uy + \int_\Gamma \mu(\mathbf{w}) \log|\mathbf{z} - \mathbf{w}|\, \mathrm{d}\Gamma_\mathbf{w} + C = 0 \quad \forall \mathbf{z} = (x, y) \in \Gamma^+,$$

$$Uy^* + \int_\Gamma \mu(\mathbf{w}) \log|\mathbf{z}^* - \mathbf{w}|\, \mathrm{d}\Gamma_\mathbf{w} + C = 0 \quad \forall \mathbf{z}^* = (x, y^*) \in \Gamma^-,$$

where $\mathbf{z}^*$ is the symmetric point of $\mathbf{z}$ with respect to x-axis. Summing up these equations, we have

$$\int_{\Gamma^+} \mu(\mathbf{w}) \log|\mathbf{z}^* - \mathbf{w}|\,|\mathbf{z} - \mathbf{w}|\, \mathrm{d}\Gamma_\mathbf{w} + \int_{\Gamma^-} \mu(\mathbf{w}^*) \log|\mathbf{z}^* - \mathbf{w}^*|\,|\mathbf{z} - \mathbf{w}^*|\, \mathrm{d}\Gamma_{\mathbf{w}^*} + 2C = 0$$

and by the symmetry and (3.24) we have

$$\int_{\Gamma^+} \mu(\mathbf{w}) \log|\mathbf{z}^* - \mathbf{w}|\,|\mathbf{z} - \mathbf{w}|\, \mathrm{d}\Gamma_\mathbf{w} + \int_{\Gamma^+} -\mu(\mathbf{w}) \log|\mathbf{z}^* - \mathbf{w}|\,|\mathbf{z} - \mathbf{w}|\, \mathrm{d}\Gamma_\mathbf{w} + 2C = 0.$$

Therefore, we have $C = 0$. Also we have $C_\varepsilon = 0$ since ρ is symmetric. Since $C_\varepsilon = C + \varepsilon C_1$, we have $C_1 = 0$.

4. Boundary modification algorithm

In this section, we propose an algorithm to find the solution shape. The essential step is to determine how to modify the boundary from a given shape. Our algorithm is an iterative procedure based on the perturbed boundary integral equations (3.15) and (3.16). We denote by $q(x)$ the given tangential velocity.

Step 1: Estimate an initial shape of body denoted by $\Gamma^{(0)}$.

Step 2: Suppose $\Gamma^{(n)}$ are previously given. In order to calculate the vortex strength $\mu^{(n)}$ located at the boundary of $\Gamma^{(n)}$, solve the direct problem:

$$\Lambda = \int_{\Gamma^{(n)}} \mu^{(n)}(\mathbf{w})\, \mathrm{d}\Gamma_\mathbf{w}, \tag{4.1}$$

$$-Uy - C = \int_{\Gamma^{(n)}} \mu^{(n)}(\mathbf{w}) \log|\mathbf{z} - \mathbf{w}|\, \mathrm{d}\Gamma_\mathbf{w} \tag{4.2}$$

for any $\mathbf{z} = (x, y) \in \Gamma^{(n)}$. Set $q^{(n)} = -2\pi\mu^{(n)}$.

Step 3: From the tangential speed $q^{(n)}$ on $\Gamma^{(n)}$, calculate the boundary variation $\rho^{(n)}$ using the perturbed boundary integral equation (3.15) in cooperation with the constant circulation condition (3.16):

$$\int_{\Gamma^{(n)}} \kappa(\mathbf{w})\rho^{(n)}(\mathbf{w})\mu^{(n)}(\mathbf{w})\,\mathrm{d}\Gamma_{\mathbf{w}} = \int_{\Gamma^{(n)}} \frac{1}{2\pi}(q - q^{(n)})(\mathbf{w})\,\mathrm{d}\Gamma_{\mathbf{w}}, \tag{4.3}$$

$$\begin{aligned} &C_1 - \pi\rho^{(n)}\mu^{(n)} - \int_{\Gamma^{(n)}} \kappa(\mathbf{w})\rho^{(n)}(\mathbf{w})\mu^{(n)}(\mathbf{w})\log|\mathbf{z}-\mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} \\ &\quad - \int_{\Gamma^{(n)}} \rho^{(n)}(\mathbf{w})\mu^{(n)}(\mathbf{w})\,\frac{(\mathbf{z}-\mathbf{w})\cdot\mathbf{n}(\mathbf{w})}{|\mathbf{z}-\mathbf{w}|^2}\,\mathrm{d}\Gamma_{\mathbf{w}} = -\int_{\Gamma^{(n)}} \frac{1}{2\pi}(q - q^{(n)})(\mathbf{w})\log|\mathbf{z}-\mathbf{w}|\,\mathrm{d}\Gamma_{\mathbf{w}} \end{aligned} \tag{4.4}$$

for any $\mathbf{z} \in \Gamma^{(n)}$.

Step 4: Modify the boundary $\Gamma^{(n)}$ toward the solution shape in the following manner:

$$\Gamma^{(n+1)}\colon\ \mathbf{w}^{(n+1)} = \mathbf{w}^{(n)} + \lambda\rho^{(n)}\mathbf{n}|_{\Gamma^{(n)}}, \tag{4.5}$$

where λ is a relaxation factor.

If $\|\rho^{(n)}\|$ is less than tolerance, then STOP.

Otherwise, go to Step 2.

As usual, the value of λ may be properly chosen for convergence.

5. Numerical experiments

5.1. Symmetric case

In our numerical experiments, the solution shape is assumed symmetric with respect to the outer flow direction, and we reformulate the $(\mathbf{n},\mathbf{s})$-coordinate system to the (x, y)-coordinate system. Hence, it suffices to consider our inverse problem (IP) on the upper half plane. The upper shape of a body is assumed to be represented by

$$\Gamma^{+}\colon\ y = f(x) \quad \text{for all } x \in [-a, a], \tag{5.1}$$

$$f(-a) = f(a) = 0. \tag{5.2}$$

The boundary variation is taken in the y-direction as in (3.29). In our algorithm, we calculate the $\mathbf{n}$ directional boundary variation ρ first and transform it to the y directional boundary variation ρ_2 by (3.30).

In the numerical calculation, the boundary element method is chosen to solve the boundary integral equations (4.2) and (4.4) (see [2]). We obtain the numerical solution by representing the body shape $y = f(x)$ by a set of N mesh points or nodes uniformly distributed along the x-axis. The unknown variables μ in direct problem (2.10) and (2.11), and ρ in inverse problem (3.15) and (3.16) are taken to be piecewise constants. Under this assumption, all integrals in the algorithm are discretized using the standard Gaussian four-point quadrature rule. Then we obtain a linear system of which solution gives the values of μ and ρ at the midpoints between the nodes. After linear interpolation

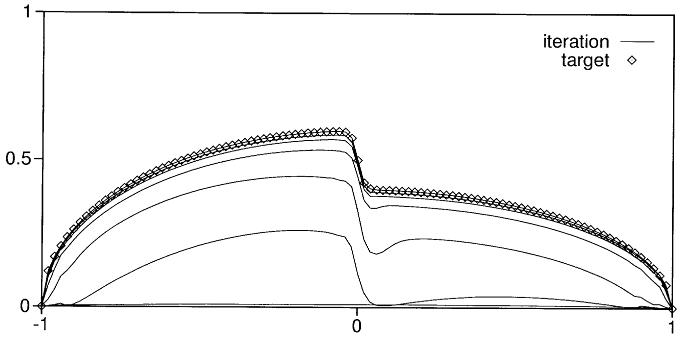

Fig. 3. Target shape: $f(x) = (-0.1 * \tanh(30x) + 0.5)\sin(\cos^{-1} x)$.

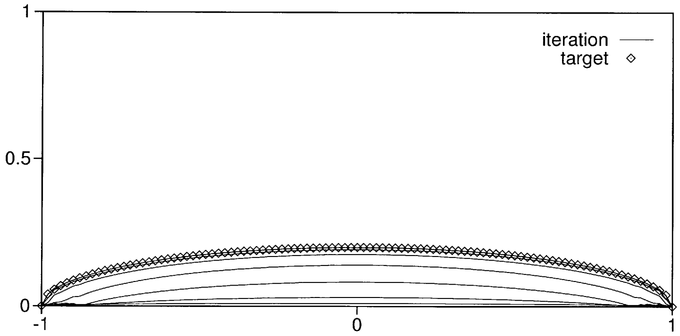

Fig. 4. Target shape: $f(x) = 0.2\sqrt{1 - x^2}$.

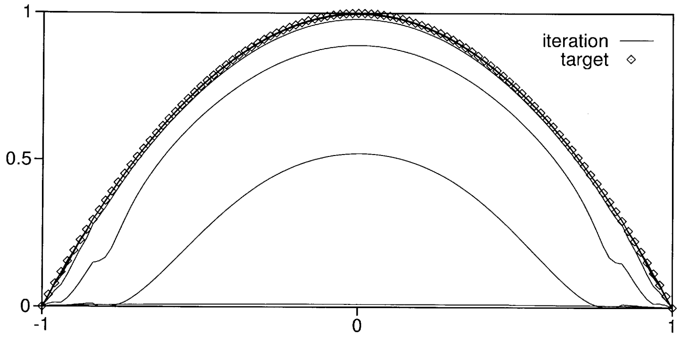

Fig. 5. Target shape: $f(x) = 1 - x^2$.

of these values back to the nodes, ρ is used for the modification of mesh points. We modify the shape in terms of the interpolated ρ at every node.

The numerical examples are presented in this section. To convince the robustness at our proposed algorithm, we consider various examples which have the following characters; steep shape (Figs. 3 and 7), smooth shape (Fig. 4), front- and rear-angled shape with nonzero curvature

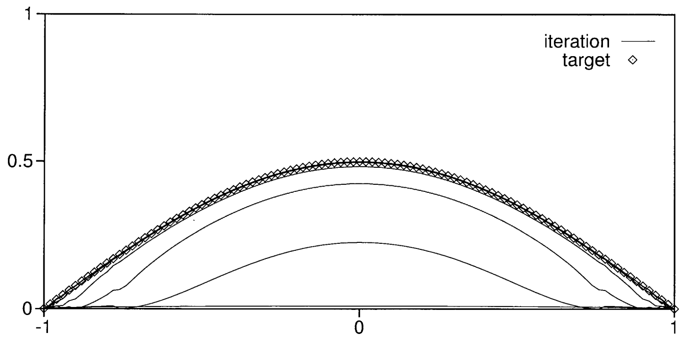

Fig. 6. Target shape: $f(x) = 0.5\cos(\frac{\pi}{2}x)$.

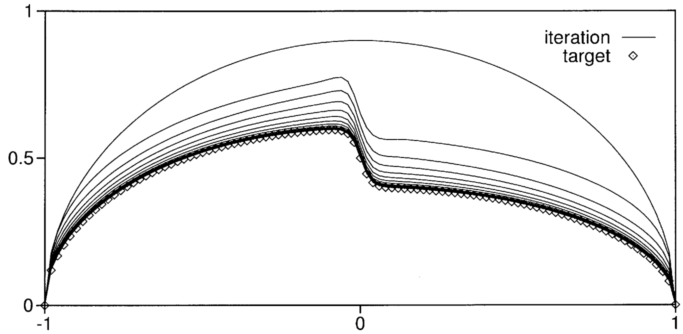

Fig. 7. Target shape: $f(x) = (-0.1 * \tanh(30x) + 0.5)\sin(\cos^{-1}x)$.

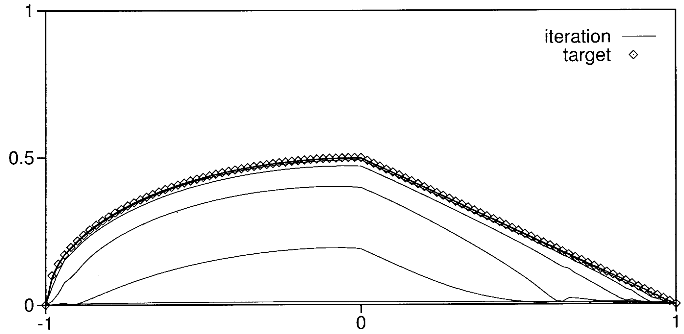

Fig. 8. Target shape: $f(x) = 0.5\sqrt{1-x^2}$ when $-1 \leqslant x \leqslant 0$, $0.5(1-x)$ when $0 \leqslant x \leqslant 1$.

(Fig. 5), front- and rear-angled shape with zero curvature (Fig. 6) and inside-angled shape (Fig. 8). The dotted lines in all figures represent the solution shapes, and the solid lines the intermediate shapes of every several iterations. The wiggles occurring at the intermediate shapes result from the nonsmooth numerical curvature effect near both ends. Although the smaller relaxation factor λ is needed when steep shape appears, λ is nearly 1 in almost all cases. This is another

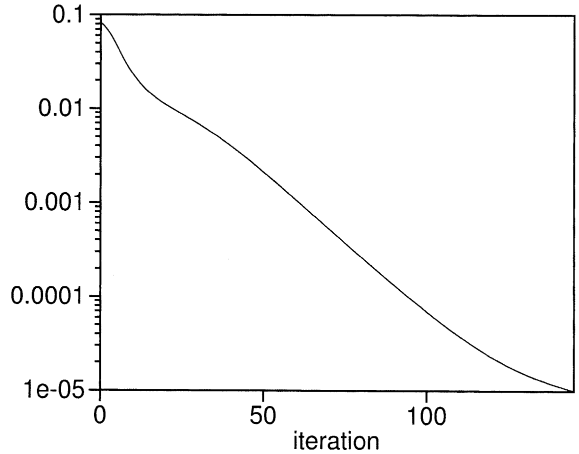

Fig. 9. L^1 norm of $\rho^{(n)}$ in the last case (Fig. 8).

evidence for the robustness of our proposed algorithm. Finally, we show the L^1 error $\|\rho^{(n)}\|$ in Fig. 9.

5.2. Asymmetric case

In this section, we illustrate an asymmetric example with distinct stagnation points on the boundary of a target body. This situation is general, interesting and important in the real case.

We assume the boundary of the target body is smooth and star-shaped with respect to the origin and a circulation Λ is properly chosen to make stagnation points on the boundary. In this case, we have to consider some extra conditions when we solve the perturbed boundary integral equations for each iteration. Indeed, if the previous boundary of a body $\Gamma^{(n)}$ is given, the solution $\rho^{(n)}\mu^{(n)}$ of the perturbed integral equations (4.3) and (4.4) has to be zero at each stagnation point on the boundary $\Gamma^{(n)}$. If $\mathbf{x}_+^{(n)}$ and $\mathbf{x}_-^{(n)}$ are distinct stagnation points on $\Gamma^{(n)}$, we have to solve the perturbed boundary integral equations (4.3) and (4.4) in Step 3 with the additional conditions

$$\rho^{(n)}(\mathbf{x}_+^{(n)})\mu(\mathbf{x}_+^{(n)}) = \rho(\mathbf{x}_-^{(n)})\mu(\mathbf{x}_-^{(n)}) = 0. \tag{5.3}$$

Reviewing the symmetric flow case, there exist always two distinct stagnation points at which the boundary of the symmetric body intersects its axis of symmetry.

Then, to calculate $\rho^{(n)}$, we divide $\rho^{(n)}\mu^{(n)}$ by $\mu^{(n)}$ in this step. However, since the numerical stagnation points contain numerical errors, we used the Fourier approximation of calculated $\rho^{(n)}$ instead of $\rho^{(n)}$ itself for boundary modification:

$$\rho^{(n)}(\theta) \approx c_0 + \sum_{k=1}^{N} s_k \sin k\theta + c_k \cos k\theta,$$

where N is the number of Fourier modes and θ $(-\pi < \theta < \pi)$ is the angle of point on $\Gamma^{(n)}$.

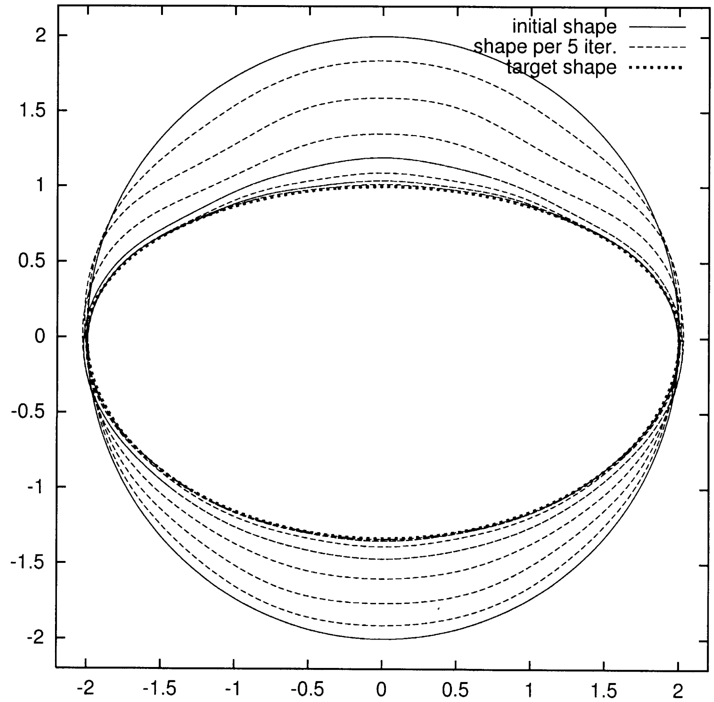

Fig. 10. Intermediate shapes toward the target shape Γ^{TG}.

Here, we choose the asymmetric target shape Γ^{TG} by

$$\Gamma^{\mathrm{TG}}: \begin{cases} \dfrac{x^2}{2^2} + \dfrac{y^2}{(\frac{4}{3})^2} = 1, \ y < 0, \\ \dfrac{x^2}{2^2} + y^2 = 1, \quad y \geqslant 0. \end{cases} \tag{5.4}$$

We also choose the circulation $\Lambda = 15$. In this case, there are two stagnation points on Γ^{TG}. From the initial shape $\Gamma^{(0)}$, the circle centered at origin with radius 2, we obtain the intermediate shapes toward the target shape Γ^{TG} as shown in Fig. 10. Here we used nine Fourier modes to approximate $\rho^{(n)}$. To calculate the solution of the Eqs. (4.3), (4.4) and (5.3), we use the constant boundary element method. To obtain the tangential velocity and the circulation more accurately on the target shape, we use smaller size elements on the target shape than those on the initial shape. In our example, we use 256 boundary elements on the initial and all intermediate shapes, and 512 elements on the target shape. Initial shape is in general far from the target shape, so that, for each initial iteration, it is possible for the modified shape to cross itself when it moves toward the target shape by poor $\rho^{(n)}$. To prevent this phenomenon, we vary the relaxation factor λ in Step 4 by

$$\lambda_n = \frac{\varepsilon_0 + (\alpha n)^2}{1 + (\alpha n)^2},$$

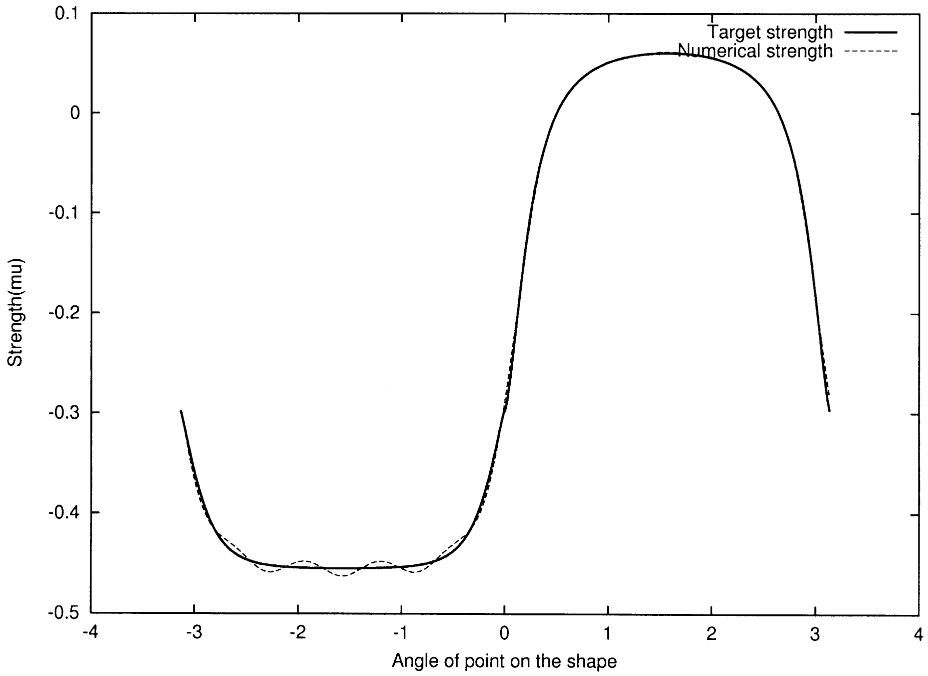

Fig. 11. Comparison μ^{TG} with μ^{NS}.

where n is the iteration number, and ε_0 and α are control parameters. In our example, we choose $\varepsilon_0 = 0.01$ and $\alpha = 0.025$.

Our algorithm converges well to the shape almost close to our target shape. We compare the strength μ^{NS} on the final shape obtained by our algorithm with μ^{TG} on the target shape. Fig. 11 shows our algorithm works well in the case of asymmetric, too.

Acknowledgements

The authors would like to say thanks to the referees for their helpful comments.

Appendix A. Determining the constant *C* in (2.11)

We show how to determine the constant C for the given circulation Λ for readers' convenience even though it appears in literature [6].

Let Γ be the boundary of a body in $\mathbb{R}^2$. If we define the harmonic function Ψ such that

$$\Psi(\mathbf{z}) = \int_\Gamma \mu(\mathbf{w}) \log |\mathbf{z} - \mathbf{w}| \, \mathrm{d}\Gamma_{\mathbf{w}},$$

then it is known (see [7]) that there exist unique $\mu(\mathbf{w})$ satisfying

$$\int_\Gamma \mu(\mathbf{w})\, \mathrm{d}\Gamma_{\mathbf{w}} = 1, \tag{A.1}$$

$$\Psi|_\Gamma = \alpha, \tag{A.2}$$

$$\Psi(\mathbf{z}) = \log|\mathbf{z}| + \mathrm{O}\left(\frac{1}{|\mathbf{z}|}\right) \quad \text{as } |\mathbf{z}| \to \infty. \tag{A.3}$$

The positive number e^α is called the outer mapping radius of Γ (see [5]). In order to determine the constant C in (2.11), we consider two separate problems as follows:

Zero circulation problem (*ZCP*). Find $\mu_0(\mathbf{w})$ and C_0 satisfying

$$\int_\Gamma \mu_0(\mathbf{w}) \log|\mathbf{z} - \mathbf{w}|\, \mathrm{d}\Gamma_{\mathbf{w}} = -Uy - C_0 \quad \forall \mathbf{z} \in \Gamma, \tag{A.4}$$

$$\int_\Gamma \mu_0(\mathbf{w})\, \mathrm{d}\Gamma_{\mathbf{w}} = 0. \tag{A.5}$$

Logarithmic capacity problem (*LCP*). Find $\mu_c(\mathbf{w})$ and α satisfying

$$\int_\Gamma \mu_c(\mathbf{w}) \log|\mathbf{z} - \mathbf{w}|\, \mathrm{d}\Gamma_{\mathbf{w}} = \alpha \quad \forall \mathbf{z} \in \Gamma, \tag{A.6}$$

$$\int_\Gamma \mu_c(\mathbf{w})\, \mathrm{d}\Gamma_{\mathbf{w}} = 1. \tag{A.7}$$

Let us consider the relationship between the pair (μ, C) and both pairs of (μ_0, C_0) and (μ_c, α). The solution pair (μ, C) in the direct problem (2.10) and (2.11) can be represented with the solution pairs (μ_0, C_0) and (μ_c, α) of (ZCP) and (LCP), respectively, such that

$$\mu = \mu_0 - \frac{\Lambda}{2\pi} \mu_c, \tag{A.8}$$

$$C = \frac{\Lambda}{2\pi} \alpha + C_0, \tag{A.9}$$

where Λ is the given circulation, α is the constant in logarithmic capacity problem (A.6) and (A.7) and C_0 is the constant to be determined by zero circulation problem (A.4) and (A.5).

Let $\mu_0(\mathbf{w})$ and $\mu_c(\mathbf{w})$ be the unique solutions of problems (A.4), (A.5) and (A.6), (A.7), respectively. Consider the single layer potential,

$$\Psi_*(\mathbf{z}) = Uy + \int_\Gamma \left(\mu_0(\mathbf{w}) - \frac{\Lambda}{2\pi} \mu_c(\mathbf{w})\right) \log|\mathbf{z} - \mathbf{w}|\, \mathrm{d}\Gamma_{\mathbf{w}}. \tag{A.10}$$

Then we have the following estimates:

$$\Psi_*|_\Gamma = -C_0 - \frac{\Lambda}{2\pi} \alpha, \tag{A.11}$$

$$\Psi_*(\mathbf{z}) = Uy - \frac{\Lambda}{2\pi} \log|\mathbf{z}| + \mathrm{O}\left(\frac{1}{|\mathbf{z}|}\right) \quad (|\mathbf{z}| \to \infty). \tag{A.12}$$

If Ψ_{**} is defined by

$$\Psi_{**}(\mathbf{z}) \equiv \Psi_{*}(\mathbf{z}) + \left(\frac{\Lambda}{2\pi}\,\alpha + C_0\right), \tag{A.13}$$

then it satisfies the following equations:

$$\Psi_{**}|_{\Gamma} = 0, \tag{A.14}$$

$$\Psi_{**}(\mathbf{z}) = Uy - \frac{\Lambda}{2\pi}\log|\mathbf{z}| + \left(\frac{\Lambda}{2\pi}\,\alpha + C_0\right) + \mathrm{O}\left(\frac{1}{|\mathbf{z}|}\right) \quad (|\mathbf{z}| \to \infty). \tag{A.15}$$

But for our solution Ψ, we can obtain the following:

$$\Psi|_{\Gamma} = 0, \tag{A.16}$$

$$\Psi(\mathbf{z}) = Uy - \frac{\Lambda}{2\pi}\log|\mathbf{z}| + C + \mathrm{O}\left(\frac{1}{|\mathbf{z}|}\right) \quad (|\mathbf{z}| \to \infty). \tag{A.17}$$

Thus, the function $\Psi_{**} - \Psi$ has zero value on Γ and

$$\lim_{|\mathbf{z}|\to\infty} \Psi_{**}(\mathbf{z}) - \Psi(\mathbf{z}) = \frac{\Lambda}{2\pi}\,\alpha + C_0 - C.$$

Since $\Psi_{**} - \Psi$ is harmonic in the exterior domain of Γ, we conclude that $C = (\Lambda/2\pi)\alpha + C_0$.

Appendix B. Example for (3.15) and (3.16)

We consider the potential flow around a cylinder in uniform flow $U\mathbf{e}_1$, $\mathbf{e}_1 = (1, 0)$. Assume the radius of cylinder is $a > 0$, and the center of the cylinder the origin. Let the shape of this cylinder be denoted by Γ. Suppose the boundary perturbation of cylinder is defined as shown in Fig. 12. Assume that Γ_ε represents the perturbed shape. For the convenience of computation, we adopt the complex notation and polar coordinates (r, θ). The complex potentials for the flows with respect to the inner circle and the outer circle are written as follows:

$$\Omega(\mathbf{z}) = U\left(\mathbf{z} + \frac{a^2}{\mathbf{z}}\right) - \mathrm{i}\,\frac{\Lambda}{2\pi}\log\frac{\mathbf{z}}{a}, \tag{B.1}$$

$$\Omega_\varepsilon(\mathbf{z}) = U\left(\mathbf{z} + \frac{(a+\varepsilon)^2}{\mathbf{z} - \mathrm{i}\varepsilon}\right) - \mathrm{i}\,\frac{\Lambda}{2\pi}\log\frac{\mathbf{z} - \mathrm{i}\varepsilon}{a + \varepsilon} - \mathrm{i}U\varepsilon. \tag{B.2}$$

Let $\mathbf{w}_\varepsilon \in \Gamma_\varepsilon$ be the perturbed point of $\mathbf{w} \in \Gamma$. Then we have

$$\mathbf{w}_\varepsilon = \mathbf{w} + \left[\varepsilon(-(1 + \sin\theta)) + \varepsilon^2\,\frac{\cos^2\theta}{2a}\right]\mathbf{n} + \mathrm{O}(\varepsilon^3)\mathbf{n} \tag{B.3}$$

and we obtain

$$\rho = -(1 + \sin\theta). \tag{B.4}$$

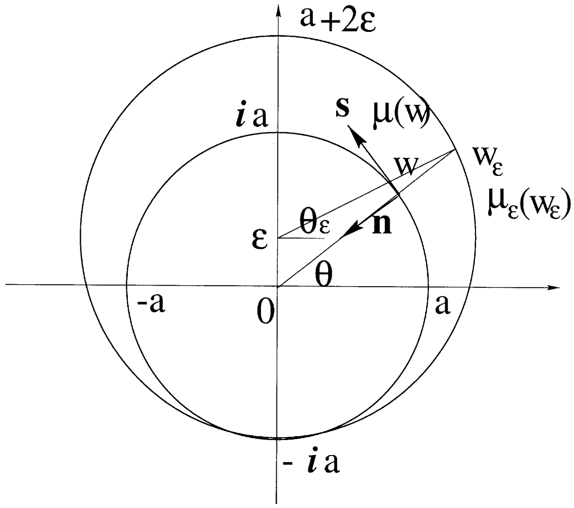

Fig. 12. Dilation and translation perturbation (y-axis).

From the identity (2.9) and the relationship between fluid velocity $\mathbf{u} = (u, v)$ and complex potential Ω

$$\frac{d\Omega}{dz} = u - iv,$$

we have

$$\mu(\mathbf{w}) = \frac{U}{\pi}\sin\theta - \frac{\Lambda}{4\pi^2 a}, \tag{B.5}$$

$$\mu_\varepsilon(\mathbf{w}_\varepsilon) = \mu(\mathbf{w}) + \varepsilon\left(-\frac{U}{\pi}\frac{\cos^2\theta}{a} + \frac{\Lambda}{4\pi^2 a^2}\right) + O(\varepsilon^2) \tag{B.6}$$

and consequently we obtain

$$\mu_1(\mathbf{w}) = -\frac{U}{\pi}\frac{\cos^2\theta}{a} + \frac{\Lambda}{4\pi^2 a^2}. \tag{B.7}$$

Also we can calculate

$$C_\varepsilon = \frac{\Lambda}{2\pi}\log a + \varepsilon\left(-U + \frac{\Lambda}{2\pi a}\right). \tag{B.8}$$

Thus, we have

$$C_1 = -U + \frac{\Lambda}{2\pi a}. \tag{B.9}$$

Now we are to calculate each term in the integral equation (3.15). From the integral identities, for any integer $n \geqslant 1$,

$$\int_{-\pi}^{\pi} \sin n\theta \log|1 - e^{i(\theta-\phi)}|\, d\theta = -\frac{\pi}{n}\sin n\phi, \tag{B.10}$$

$$\int_{-\pi}^{\pi} \cos n\theta \log|1 - e^{i(\theta-\phi)}|\, d\theta = -\frac{\pi}{n}\cos n\phi, \tag{B.11}$$

we can calculate the terms in (3.15):

$$C_1 = -U + \frac{\Lambda}{2\pi a} \tag{B.12}$$

$$-K_\Gamma[\rho\mu](\mathbf{z}) = \left(U - \frac{\Lambda}{2\pi a}\right)\log a - \left(U - \frac{\Lambda}{4\pi a}\right)\sin\phi + \frac{U}{4}\cos 2\phi, \tag{B.13}$$

$$-D_\Gamma^*[\rho\mu](\mathbf{z}) = -\frac{\Lambda}{4\pi a} + \frac{U}{2}, \tag{B.14}$$

$$F_\Gamma[\mu_1](\mathbf{z}) = -\left(U - \frac{\Lambda}{2\pi a}\right)\log a + \frac{U}{4}\cos 2\phi. \tag{B.15}$$

Therefore, we are confident in the integral equation (3.15).

Appendix C. Some remarks on nonuniqueness of ρ

Since we have to solve the integral equation (3.14) numerically, there is a need to demonstrate the uniqueness of the solution ρ for the boundary integral equation (3.14). To do this, it suffices to show that, if we assume that $\mu_1 = 0$ on Γ, the corresponding solution ρ is zero. This implies that if the solution shape is achieved, then there is no need of further changes in the boundary. Here, we consider the special case that the profile of a body is the circle with radius $a > 0$ and the target speed q is given such that

$$q(\theta) = -2U\sin\theta + \frac{\Lambda}{2\pi a}.$$

Let $L^2[-\pi,\pi]$ be the function space which is composed of all square integrable functions on the interval $[-\pi,\pi]$ equipped with the usual inner product (f,g) defined by

$$(f,g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

If we find the boundary variation ρ periodic and continuous on $[-\pi,\pi]$, then it may be expanded in terms of Fourier series such that

$$\rho(\theta) = \beta_0 + \sum_{n=1}^{\infty}\beta_n e^{in\theta} + \sum_{n=1}^{\infty}\overline{\beta_n}e^{-in\theta}, \tag{C.1}$$

where $\overline{\beta_n}$ is the complex conjugate of β_n and β_0 is real. The Riemann–Lebesgue lemma provides us the decay of coefficients

$$\lim_{n\to\infty}\beta_n = 0, \tag{C.2}$$

since continuous function on $[-\pi,\pi]$ are in $L^2[-\pi,\pi]$.

Substituting the Fourier series expansion (C.1) of ρ into the integral equation (3.14) and applying $\mu_1(\mathbf{w})=0$ and calculating each integral using the integral identities (B.10) and (B.11), we can obtain the following results:

$$
\begin{aligned}
-K_\Gamma[\mu\rho](\mathbf{z}) = &-\frac{U}{2\mathrm{i}}\,[2(\overline{\beta_1}-\beta_1)\log a] + \frac{\Lambda}{2\pi a}\,\beta_0\log a \\
&-\frac{U}{2\mathrm{i}}\left[\sum_{n=1}^{\infty}\frac{1}{n}\,(\beta_{n+1}-\beta_{n-1})\mathrm{e}^{\mathrm{i}n\phi} - \frac{1}{n}\,(\overline{\beta_{n+1}}-\overline{\beta_{n-1}})\mathrm{e}^{-\mathrm{i}n\phi}\right] \\
&-\frac{\Lambda}{4\pi a}\left[\sum_{n=1}^{\infty}\frac{1}{n}\,\beta_n\mathrm{e}^{\mathrm{i}n\phi} + \sum_{n=1}^{\infty}\frac{1}{n}\,\overline{\beta_n}\mathrm{e}^{-\mathrm{i}n\phi}\right],
\end{aligned}
$$

$$
\rho(\mathbf{z})D_\Gamma[\mu](\mathbf{z}) = \frac{\Lambda}{4\pi a}\left[\beta_0 + \sum_{n=1}^{\infty}\beta_n\mathrm{e}^{\mathrm{i}n\phi} + \sum_{n=1}^{\infty}\overline{\beta_n}\mathrm{e}^{-\mathrm{i}n\phi}\right],
$$

$$
-D_\Gamma^*[\rho\mu](\mathbf{z}) = -\frac{U}{2\mathrm{i}}\,(\overline{\beta_1}-\beta_1) + \frac{\Lambda}{4\pi a}\,\beta_0
$$

and

$$
-U\rho(\mathbf{z})(\mathbf{e}_2\cdot\mathbf{n})(\mathbf{z}) = \frac{U}{2\mathrm{i}}\,[(\overline{\beta_1}-\beta_1)] + \frac{U}{2\mathrm{i}}\left[\sum_{n=1}^{\infty}(\beta_{n-1}-\beta_{n+1})\mathrm{e}^{\mathrm{i}n\phi} + (\overline{\beta_{n+1}}-\overline{\beta_{n-1}})\mathrm{e}^{-\mathrm{i}n\phi}\right].
$$

Equating both sides of (3.14) and rearranging the resultant equation in each Fourier term, the following equation can be obtained:

$$
\begin{aligned}
&-C_1 + \frac{U}{2\mathrm{i}}\,(\overline{\beta_1}-\beta_1) + \frac{U}{2\mathrm{i}}\left[\sum_{n=1}^{\infty}(\beta_{n-1}-\beta_{n+1})\mathrm{e}^{\mathrm{i}n\phi} - \sum_{n=1}^{\infty}(\overline{\beta_{n-1}}-\overline{\beta_{n+1}})\mathrm{e}^{-\mathrm{i}n\phi}\right] \\
&\quad = \frac{\Lambda}{2\pi a}\,\beta_0(1+\log a) - \frac{U}{2\mathrm{i}}\,(\overline{\beta_1}-\beta_1)(1+2\log a) + \frac{U}{2\mathrm{i}}\,[(\beta_0-\beta_2)\mathrm{e}^{\mathrm{i}\phi} - (\overline{\beta_0}-\overline{\beta_2})\mathrm{e}^{-\mathrm{i}\phi}] \\
&\qquad + \sum_{n=2}^{\infty}\left[\frac{U}{2\mathrm{i}}\,(\beta_{n-1}-\beta_{n+1})\,\frac{1}{n} - \frac{\Lambda}{4\pi a}\,\beta_n\left(\frac{1}{n}-1\right)\right]\mathrm{e}^{\mathrm{i}n\phi} \\
&\qquad + \sum_{n=2}^{\infty}\left[-\frac{U}{2\mathrm{i}}\,(\overline{\beta_{n-1}}-\overline{\beta_{n+1}})\,\frac{1}{n} - \frac{\Lambda}{4\pi a}\,\overline{\beta_n}\left(\frac{1}{n}-1\right)\right]\mathrm{e}^{-\mathrm{i}n\phi}.
\end{aligned}
$$

Comparing the coefficients between both Fourier series, we can obtain the recursive relations:

$$
-\,C_1 + (1+\log a)\left[-\frac{\Lambda}{2\pi a}\,\beta_0 + \frac{U}{\mathrm{i}}\,(\overline{\beta_1}-\beta_1)\right] = 0, \tag{C.3}
$$

$$\beta_{n-1} - \beta_{n+1} - \mathrm{i}\,\frac{\Lambda}{2\pi a U}\,\beta_n = 0, \tag{C.4}$$

$$\overline{\beta_{n-1}} - \overline{\beta_{n+1}} + \mathrm{i}\,\frac{\Lambda}{2\pi a U}\,\overline{\beta_n} = 0, \tag{C.5}$$

for all $n \geqslant 2$.

Applying the constant circulation condition (3.16) to ρ, we have

$$\frac{U}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \overline{\beta_1} - \beta_1 \,\mathrm{d}\theta - \int_{-\pi}^{\pi} \frac{\Lambda}{4\pi^2 a}\,\beta_0 \,\mathrm{d}\theta = 0. \tag{C.6}$$

This implies that

$$\frac{U}{\mathrm{i}}\,(\overline{\beta_1} - \beta_1) - \frac{\Lambda}{2\pi a}\,\beta_0 = 0. \tag{C.7}$$

Thus, we obtain $C_1 = 0$. And the general term β_n of Fourier coefficients recursively defined in (5.40) is

$$\beta_n = \beta_1\,\frac{a_+^{n-2} - a_-^{n-2}}{a_+ - a_-} + \beta_2\,\frac{a_+^{n-1} - a_-^{n-1}}{a_+ - a_-}, \quad n \geqslant 1, \tag{C.8}$$

where $a_+ = -\mathrm{i}K + \sqrt{1-K^2}$, $a_- = -\mathrm{i}K - \sqrt{1-K^2}$ and $K = \Lambda/4\pi a U$.

Here if we assume $\Lambda = 0$, then we have $a_+ = 1$ and $a_- = -1$. From the general term (C.8), we can conclude that

$$\beta_{2n} = \beta_2, \quad \beta_{2n-1} = \beta_1, \qquad n \geqslant 1. \tag{C.9}$$

In order to satisfy $\rho \in L^2[-\pi, \pi]$, the Fourier coefficients must satisfy the decay condition (C.2), so that we have $\beta_1 = \beta_2 = 0$.

Therefore, we obtain the following consequence:

$$\rho(\theta) = \beta_0.$$

To ensure the uniqueness of ρ, one extra restriction is needed. For this, we fix one point on shape, i.e., $\rho(0) = 0$. Then we have

$$\beta_0 = 0.$$

We have discussed the uniqueness of perturbed boundary integral equations in the case that our goal shape is a circle under several assumptions. The above discussion gives us a guideline about the unique solvability.

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