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MAXIMUM PRINCIPLE AND CONVERGENCE ANALYSIS FOR THE MESHFREE POINT COLLOCATION METHOD*

DO WAN KIM† AND WING KAM LIU‡

Abstract. The discrete Laplacian operator is considered in the sense of the meshfree point collocation method which will be called the strong meshfree Laplacian operator. To define the strong meshfree Laplacian operator, we use the fast version of the generalized moving least square approximation, which can calculate the approximated derivatives of shape functions. Some types of the locally layered node distribution are defined in this paper, and two specific domains are constructed onto which we can distribute locally layered nodes. At such nodes, the discrete maximum principle can be shown to hold through the representation formula for the strong meshfree Laplacian operator. The discrete maximum principle, together with the reproducing property of the meshfree approximations, results in an a priori estimate for the strong meshfree Laplacian operator in the nodal solution space. Furthermore, the a priori estimate we have obtained guarantees the existence and the uniqueness of the numerical solution and plays a central role in achieving converged results for the Poisson problem with Dirichlet boundary conditions in the nodal solution space. The order of convergence of the nodal solutions can be raised up to $O(h^2)$ at the proposed type of nodes in specific domains. For generally shaped domains immersed in the previously mentioned domains, we can obtain the first order convergence result of $O(h)$.

Key words. strong meshfree Laplacian operator, discrete maximum principle, a priori estimate, meshfree point collocation method, generalized moving least square approximation, convergence analysis

AMS subject classifications. 65D25, 65M15, 65M12, 65M70

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1. Introduction.

In the field of numerical computations, meshfree methods have been developed for more than a decade. In order to solve many physical problems represented by partial differential equations, researchers and scientists have proposed meshfree approximations, examples of which include the element free Galerkin method [3], the moving least square reproducing kernel method [10, 17], the partition of unity finite element method [2], the reproducing kernel hierarchical partition of unity [11, 12, 13, 15], the reproducing kernel element method [14, 16, 18, 20], etc.

The above pioneering work has presented a common framework for meshfree methodologies and shown the potential of meshfree methods. In many cases, the work in meshfree fields has been based on the weak formulation of the model equation, but only a few papers supply the mathematical convergence for numerical solutions in the one-dimensional (1-D) case [2, 12].

In this paper, we focus on uniform convergence analysis for the numerical solution of the strong formulation using a meshfree approximation. Here we use the generalized moving least square approximation for efficient calculation of higher order

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†Department of Applied Mathematics, College of Science and Technology, Hanyang University, Ansan, Kyeonggi 426-791, Republic of Korea. Visiting Scholar, Department of Mechanical Engineering, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208-3111 (d-kiml@northwestern.edu).

‡Corresponding author. Department of Mechanical Engineering, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208-3111 (w-liu@northwestern.edu).

shape function derivatives, which stem from the reproducing kernel hierarchical partition of unity method by Li and Liu [11, 12], and from the fast moving least square reproducing kernel approximation method by D. W. Kim and Y. Kim [8, 9]. Such approximations are desired to convert the higher order differential operator into a discrete one by attacking the strong formulation and utilizing the point collocation method. The meshfree point collocation method (MPCM) follows the philosophy of the meshfree method in which no structured meshes are used.

In mathematical analyses of the Galerkin formulation using meshfree approximations, difficulties arise mainly in the treatment of Dirichlet boundary conditions. The construction of a test function space which belongs to the Sobolev space $H_0^1(\Omega)$ is a challenging issue in meshfree Galerkin formulations, particularly for higher dimensions. However, once one can overcome this difficulty, then the remaining part of the convergence analysis follows similarly from the mathematical theories of the finite element method. For the finite element method, uniform convergence has been shown by Ciarlet and Raviart [4] for second order elliptic models under some specially shaped meshes. To achieve first order uniform convergence, they used the discrete maximum principle and obtained an a priori estimate for the discrete solution of the Poisson-type problem. For local pointwise error estimates in finite element methods, one can see the important results for second order elliptic problems in [5] written by L. B. Wahlbin.

The meshfree point collocation methods, in contrast to the Galerkin formulation, have few mathematical results, as the theory of function spaces is not directly available. Thus, the objective of this paper is to build the underlying theories for the MPCM, particularly for the discrete Laplacian operator, and based on those theories to prove uniform convergence of the nodal solutions of the Poisson problem with Dirichlet boundary conditions. For the convergence estimate in the MPCM, the first step is to define the rigorous point collocation scheme—an important portion of the mathematical analysis. Next, we will show that the discrete Laplacian operator satisfies the discrete maximum principle for some classes of nodes, and then obtain an a priori estimate for the strong meshfree Laplacian operator on the nodal solution space, provided the discrete maximum principle holds.

As for the discrete maximum principle itself, many researchers are interested in cases in which it occurs and their applications [1, 4, 7, 19, 21, 22]. The discrete maximum principle for the discretized Laplacian operator in the finite difference method on evenly spaced grid points is well-known and is closely related to the mean-value property for the Laplace solutions. This means that the average value on the surrounding four points in a five-point stencil for the Laplacian operator is equal to the center value. Inspired by the difference scheme for the Laplacian operator in the finite difference method, we can obtain the representation formula at each node for the strong meshfree Laplacian operator which is followed by the discrete maximum principle.

As a result of the discrete maximum principle, an a priori estimate for the strong meshfree Laplacian is derived in the nodal solution space. The a priori estimate guarantees the existence and the uniqueness of the numerical solution governed by the point collocation scheme. We finally achieve convergence for the numerical solutions of the Poisson problem with Dirichlet boundary conditions. The convergence order can be up to second order on some specific domains, while we have first order convergence for general domains immersed in the specific domains.

We know that finite difference methods and finite element methods have discrete maximum principle for elliptic partial differential equations. However, for meshfree strong form collocation methods, the authors are not aware of any previous theoretical results. This is an important aspect, and this is the first paper to the authors knowledge that deals with the theoretical foundation of the meshfree collocation method.

2. Generalized moving least square reproducing kernel approximation. To make this paper self-contained, we will describe how to obtain the meshfree approximation of the Laplacian operator. For a moment, we will make general statements on the moving least square reproducing kernel approximation as we can see similar content in the literature [8, 11, 12].

Let Ω be a bounded domain in $\mathbb{R}^n$ and also $\Lambda \equiv \{\mathbf{x}_I \in \overline{\Omega} \,|\, I = 1, \ldots, N\}$ where Λ is a set of distributed nodes in $\overline{\Omega}$. Throughout the paper, the multi-index notation and related definitions are employed as follows:

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad |\alpha| \equiv \sum_{i=1}^{n} \alpha_i, \quad \alpha! \equiv \alpha_1!\alpha_2! \cdots \alpha_n!, \tag{2.1}$$

$$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad \mathbf{x}^\alpha \equiv x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \quad D_{\mathbf{x}}^\alpha \equiv \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}, \tag{2.2}$$

where α_k's are nonnegative integers and α is called the multi-index. We consider a vector of complete basis functions of order m in $\mathbb{R}^n$ such that

$$\mathbf{B}_m(\mathbf{x}) = (b_{\beta_1}(\mathbf{x}), b_{\beta_2}(\mathbf{x}), \ldots, b_{\beta_L}(\mathbf{x}))^T, \quad |\beta_k| \le m, \tag{2.3}$$

where β_k's are all multi-indices in lexicographical order. Here we note that the number of β_k's is $L \equiv \frac{(m+n)!}{m!n!}$ and the complete basis of order m means that the $L \times L$ matrix $J_{\mathbf{B}_m}(\mathbf{0})$ is invertible if we define the Jacobian of $\mathbf{B}_m(\mathbf{x})$ at $\mathbf{0}$ as

$$J_{\mathbf{B}_m}(\mathbf{0}) \equiv \lim_{\mathbf{x}\to\mathbf{0}} \left(D_{\mathbf{x}}^\alpha \, b_\beta(\mathbf{x})\right), \quad |\alpha|, |\beta| \le L. \tag{2.4}$$

Let $B_r(\mathbf{z}) \equiv \{\mathbf{y} \,|\, \|\mathbf{y}-\mathbf{z}\| < r\}$ be the r-ball in $\mathbb{R}^n$ with center $\mathbf{z}$. We introduce the continuous nonnegative window function with its support on $\overline{B_1(\mathbf{0})}$ of the following type

$$W(\mathbf{x}) = (1 - \|\mathbf{x}\|^{\frac{1}{2}})^2 \quad \text{for} \quad \|\mathbf{x}\| < 1, \, \mathbf{x} \in \mathbb{R}^n \tag{2.5}$$

and the continuous positive dilation function

$$\rho(\mathbf{x}) > 0 \quad \text{on} \quad \overline{\Omega}. \tag{2.6}$$

For brevity, we will use $\rho_{\mathbf{x}}$ instead of $\rho(\mathbf{x})$.

Remark 1. The decreasing rate of the window function values apart from the origin is essential in proving the discrete maximum principle for the strong meshfree Laplacian operator introduced in a later section. The window function of the form (2.5) meets this decreasing rate. The support of the window function in this paper has the n-dimensional unit ball shape.

Remark 2. The dilation parameter used in most meshfree methods can be replaced with the dilation function defined on the whole domain $\overline{\Omega}$. The required regularity for the dilation function is only the continuity to be well defined when the center of window function moves to the evaluation point. The dilation function controls the support and its size of shape functions, and thus is directly available to the geometrically multiple scale problems [9].

The subsequent procedure to make the shape functions and the approximated derivative operators is addressed in detail in Appendix I. This includes the generalized reproducing properties of meshfree shape functions and the proposal of a sufficient condition to regenerate the dilated basis functions. These are novel differences from the standard moving least square reproducing kernel approximation.

From this point forward, we restrict our attention to polynomial basis functions; that is, if there is no comment, then the basis functions will be maintained as complete polynomials up to order m

$$\mathbf{B}_m(\mathbf{x}) = (\mathbf{x}^{\beta_1}, \mathbf{x}^{\beta_2}, \ldots, \mathbf{x}^{\beta_L}), \quad |\beta_k| \le m \tag{2.7}$$

throughout the mathematical analysis.

For the subsequent analysis, we require the definition of the proper node distributions.

DEFINITION 1 (proper triple). *Let* $(\Omega, \Lambda, \rho_{\mathbf{x}})$ *be the triple of a domain, a set of distributed nodes on* $\overline{\Omega}$*, and a dilation function. The triple* $(\Omega, \Lambda, \rho_{\mathbf{x}})$ *is said to be proper if the moment matrix* $M^{\rho_{\mathbf{x}}}(\mathbf{x}_I)$ *is invertible for every interior node* $\mathbf{x}_I \in \Lambda \cap \Omega$ *under the dilation function* $\rho_{\mathbf{x}}$.

This definition is preventive of the degenerate distribution of nodes to approximate functions in the meshfree method.

3. Problem statement and the definition of the discrete problem.

We will now consider the discretization of the Poisson problem as the popular model in the second order elliptic problem with Dirichlet boundary conditions and prepare the terminology for its convergence analysis. The Poisson equation uses the Laplacian operator, the principal operator in most physical models. Furthermore, the Laplacian is an interesting operator in itself, since it has the salient feature referred to as the maximum principle. Many mathematical theories have been developed based on this property. Among them, the regularity and the uniqueness of solutions of the Poisson equation is highly involved with the maximum principle. For the discrete case analogous to the continuous one, the discrete maximum principle has been reported not only for the Galerkin formulation [4] in the finite element method but also for the solution of some algebraic systems [7].

We consider the Poisson problem with Dirichlet data on the boundary of a domain Ω and propose the corresponding discrete problem using the point collocation approach based on the generalized meshfree approximation operators described in the previous section and Appendix I in detail. The model problem considered in this paper is governed by the following equations:

$$(\mathbf{CP}) \begin{cases} \Delta u = f, & \text{in } \Omega \\ u = g, & \text{on } \Gamma, \end{cases} \tag{3.1}$$

where $\Gamma \equiv \partial\Omega$ represents the boundary of the open bounded domain Ω. According to Theorem 6.13 in [6] for the general existence and regularity of a unique solution of $(\mathbf{CP})$, if Ω is a bounded domain satisfying an exterior sphere condition at every boundary point and we have $f \in C^{s-2,\alpha}(\Omega)$ for $s = 3, 4$ and $g \in C(\partial\Omega)$, then the Dirichlet problem $(\mathbf{CP})$ has a unique solution $u \in C^0(\overline{\Omega}) \cap C^{s,\alpha}(\Omega)$, where $C^0(\overline{\Omega})$ is the vector space to consist of all bounded and uniformly continuous functions on Ω and $C^{s,\alpha}(\Omega)$ represents the Hölder space of exponent $0 < \alpha \le 1$ equipped with the norm

$$\|v\|_{C^{s,\alpha}(\Omega)} \equiv \max_{0 \le |\beta| \le s} \sup_{\mathbf{x} \in \Omega} |D^\beta v(\mathbf{x})| + \max_{0 \le |\beta| \le s} \sup_{\mathbf{x},\mathbf{y} \in \Omega, \mathbf{x} \ne \mathbf{y}} \frac{|D^\beta v(\mathbf{x}) - D^\beta v(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha}. \tag{3.2}$$

To discretize the problem (**CP**) in terms of meshfree point collocation method, we focus on the second order meshfree approximation ($m = 2$); that is, the complete polynomial basis functions up to second order are adopted to obtain all the shape functions (see Appendix I). It is taken to satisfy the minimum order of consistency for the discretization of second order partial differential equations since we approximate the Laplace operator in a pointwise manner. Higher order approximation could be better than the second order one in general but, since the focus in this paper is on analyzing the structure of the meshfree Laplace operator, the second order meshfree approximation must be the starting point. We also consider the 2-D space ($n = 2$) and hence the relevant multi-index β_k ($k = 1, \ldots, 6$) appearing in the basis polynomials (2.7) which are fixed in lexicographical order as follows:

$$(0,0), \quad (1,0), \quad (0,1), \quad (2,0), \quad (1,1), \quad (0,2). \tag{3.3}$$

The analysis in this paper is expected to hold for the higher dimensions as well as for higher orders. It could depend on the construction of the local nodes and the adequate dilation function.

In order to define the discrete counterpart of the continuous problem (3.1), we assume Λ is a set of well distributed nodes on the domain Ω and its boundary, so that $(\Omega, \Lambda, \rho_{\mathbf{x}})$ becomes the proper triple. Let $C(\overline{\Omega})$ be the space of continuous functions up to the boundary of Ω, and V be the finite dimensional space of functions defined on Λ. We will call the function space V the nodal solution space if V is equipped with the following seminorm:

$$\|v\|_{\infty,A} \equiv \max_{\mathbf{x}_K \in A} |v_K| \quad \text{when} \quad v \in V, \tag{3.4}$$

where A is a nonempty subset of Λ. In the case when $A = \Lambda$, the seminorm becomes the norm on V.

If the restriction map $i : C(\overline{\Omega}) \to V$ to Λ is defined such that, for any $u \in C(\overline{\Omega})$,

$$i(u)(\mathbf{x}_I) \equiv u(\mathbf{x}_I) \quad \text{for any } \mathbf{x}_I \in \Lambda, \tag{3.5}$$

then the point collocation Laplacian operator Δ^ρ can be defined on V into itself such that if $v \in V$, then

$$(\Delta^\rho v)(\mathbf{x}_I) \equiv \sum_{\mathbf{x}_J \in \Lambda} v(\mathbf{x}_J)\, \psi_J^{\Delta}(\mathbf{x}_I) \quad \text{for any } \mathbf{x}_I \in \Lambda, \tag{3.6}$$

where the function $\psi_J^{\Delta}(\mathbf{x}_I)$ will be called the Laplacian shape function at $\mathbf{x}_I$ and is defined by

$$\psi_J^{\Delta}(\mathbf{x}_I) \equiv \psi_J^{[(2,0)]}(\mathbf{x}_I) + \psi_J^{[(0,2)]}(\mathbf{x}_I) \tag{3.7}$$

which is the sum of the $(2,0)$th and $(0,2)$th approximate derivatives whose definition comes from (7.3) in Appendix I. In fact, the operator Δ^ρ stems from the meshfree approximated Laplacian operators $D_{m,\rho_{\mathbf{x}}}^{(2,0)} + D_{m,\rho_{\mathbf{x}}}^{(0,2)} \sim \Delta$. Hereafter, we will often use the symbol u_J instead of $u(\mathbf{x}_J)$ if $u \in V$ and $\mathbf{x}_J \in \Lambda$.

Using these operators i and Δ^ρ, we define the meshfree point collocation discretization of Poisson problem (**CP**) as the following:

$$u_h \in V \ : \begin{cases} \Delta^\rho u_h = i(f), & \text{on } \Lambda^o \\ u_h = g, & \text{on } \Lambda^b \end{cases}, \tag{3.8}$$

where $\Lambda = \Lambda^o \cup \Lambda^b$ and Λ^o and Λ^b are sets of interior nodes and Dirichlet boundary nodes, respectively. Consequently, our discrete problem for (**CP**) results in finding the nodal solution $u_h \in V$ such that

$$(3.9)\qquad (\mathbf{DP}) \begin{cases} u_h \in V_g \equiv \{v_J \in \mathbb{R} \,|\, v_K = g(\mathbf{x}_K) \text{ for all } \mathbf{x}_K \in \Lambda^b\} \subset V \\ \Delta^\rho u_h = i(f), \quad \text{on } \Lambda^o. \end{cases}$$

This final formulation will be called *the discrete Poisson problem* (**DP**) and the operator Δ^ρ will be called *the strong meshfree Laplacian operator*.

In order to attain the error estimate, we begin discussion of the discrete maximum principle for the strong meshfree Laplacian operator Δ^ρ.

4. Discrete maximum principle for the strong meshfree Laplacian operator $\boldsymbol{\Delta^\rho}$. Let $(\Omega, \Lambda, \rho_{\mathbf{x}})$ be the proper triple. For convenience sake, the r-neighbor nodes of $\mathbf{x}$ are assumed to be the following set:

$$(4.1)\qquad \Lambda_r(\mathbf{x}) \equiv \{\mathbf{x}_K \in \Lambda \,|\, \mathbf{x}_K \in B_r(\mathbf{x})\}, \quad r > 0$$

and the symbol A^* for a subset $A \subset \Lambda$ implies the set defined by

$$(4.2)\qquad A^* \equiv \bigcup_{\mathbf{x}_J \in A} \Lambda_{\rho_{\mathbf{x}_J}}(\mathbf{x}_J).$$

If there is no confusion, we briefly write $\Lambda(\mathbf{x}_K)$ instead of $\Lambda_{\rho_{\mathbf{x}_K}}(\mathbf{x}_K)$ for any node $\mathbf{x}_K \in \Lambda$.

We now state the definition of the discrete maximum principle.

Definition 2 (discrete maximum principle for the operator Δ^ρ). *Assume the proper triple $(\Omega, \Lambda, \rho_{\mathbf{x}})$ is given. We will say the strong meshfree Laplacian Δ^ρ satisfies the discrete maximum principle at a node $\mathbf{x}_I \in \Lambda$ if the condition $(\Delta^\rho v)(\mathbf{x}_I) \geq 0$ for $v \in V$ implies that either $v_I < \max_{\mathbf{x}_K \in \Lambda(\mathbf{x}_I) \setminus \{\mathbf{x}_I\}} v_K$ or $v_K = v_I$ for all $\mathbf{x}_K \in \Lambda(\mathbf{x}_I)$. We also will say the operator Δ^ρ satisfies the discrete maximum principle on a subset $A \subset \Lambda$ if it satisfies the discrete maximum principle at all nodes in A.*

In fact, the discrete maximum principle for the discrete Laplace operator is known to depend on the geometry of the mesh in the finite element method and the orthogonal grid in the finite difference method, respectively. For example, if all the angles of the triangles of the triangulation on a domain are less than or equal to $\frac{\pi}{2}$, then the discrete maximum principle is known to hold in the finite element method [4]. Hence it can also be expected that the relative attitude between nodes strongly affects this kind of phenomenon in the meshfree area. Therefore, we are interested in finding such node distributions from the meshfree point of view. On the other hand, to perform the convergence analysis on such nodes, we have to inspect closely the moment matrix and its inverse, since it is located in the core of Laplacian shape functions in (3.7).

The moment matrix for the given set Λ of nodes has the following form in the generalized moving least square reproducing kernel approximation [17] (see also (7.5) in Appendix I)

$$(4.3)\qquad M^{\rho_{\mathbf{x}}}(\mathbf{x}) = \sum_{\mathbf{x}_I \in \Lambda} \mathbf{B}_m\left(\frac{\mathbf{x}_I - \mathbf{x}}{\rho_{\mathbf{x}}}\right)\mathbf{B}_m^T\left(\frac{\mathbf{x}_I - \mathbf{x}}{\rho_{\mathbf{x}}}\right) W\left(\frac{\mathbf{x}_I - \mathbf{x}}{\rho_{\mathbf{x}}}\right),$$

where $\mathbf{B}_m\left(\frac{\mathbf{y}-\mathbf{x}}{\rho_{\mathbf{x}}}\right)$ is the normalized basis polynomial up to order m at the center

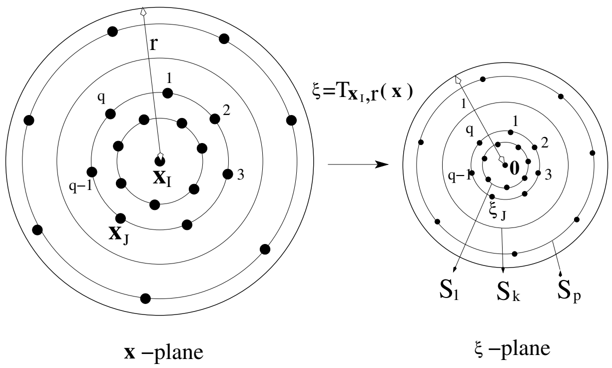

FIG. 4.1. *The locally (p, q)-layered nodes ($\mathbf{x}$-plane) and the normalizing ones (ξ-plane) by $T_{\mathbf{x}_I,r}(\mathbf{x})$. S_K's are the layers of the normalized nodes.*

point $\mathbf{x} \in B_{\rho_\mathbf{x}}(\mathbf{x})$ such that

$$\mathbf{B}_m\left(\frac{\mathbf{y}-\mathbf{x}}{\rho_\mathbf{x}}\right) = \left(\left(\frac{\mathbf{y}-\mathbf{x}}{\rho_\mathbf{x}}\right)^{\beta_1}, \left(\frac{\mathbf{y}-\mathbf{x}}{\rho_\mathbf{x}}\right)^{\beta_2}, \ldots, \left(\frac{\mathbf{y}-\mathbf{x}}{\rho_\mathbf{x}}\right)^{\beta_L}\right), \quad |\beta_k| \le m. \tag{4.4}$$

To calculate the moment matrix and its inverse concretely, we need to focus on some class of node distributions. In order to define some classes of nodes, we must first introduce the normalizing transformation $T_{\mathbf{x},r}(\mathbf{y}) : B_r(\mathbf{x}) \to B_1(\mathbf{0})$ such that, as shown in Figure 4.1,

$$\xi = T_{\mathbf{x},r}(\mathbf{y}) \equiv \frac{\mathbf{y}-\mathbf{x}}{r}. \tag{4.5}$$

DEFINITION 3 (layered node distribution). *Let $A_r(\mathbf{x}_I) \equiv \{\mathbf{x}_K \mid \mathbf{x}_K \in B_r(\mathbf{x}_I)\}$ be the finite subset of nodes within the distance r around $\mathbf{x}_I$. The set of nodes $A_r(\mathbf{x}_I)$ is said to be the locally (p, q)-layered at $\mathbf{x}_I$ if all the normalized nodes in $T_{\mathbf{x}_I,r}(A_r(\mathbf{x}_I))$ remain on the p-layer sets $S_1, \ldots, S_p$ in the increasing radial direction from the origin and the q nodes are distributed evenly on each layer. All the layer sets S_k's have the spherical shape only. Furthermore, we say that the node set Λ is possibly layered if, for any interior node $\mathbf{x}_I \in \Lambda$, $\Lambda(\mathbf{x}_I)$ is the locally (p, q)-layered at $\mathbf{x}_I$ for some $p, q > 0$.*

As a matter of fact, the possibly layered distribution of nodes is not a simple matter since the property of the locally (p, q)-layered at every neighboring node has to be achieved. Thus, we will propose two kinds of available distribution of nodes and show that they are the possibly layered. On such types of the possibly layered nodes, the discrete maximum principle for the strong meshfree Laplacian will be proven.

We begin with the calculation of the moment matrix that will play an essential role in proving the discrete maximum principle on some possibly layered nodes. If the subset of nodes $\Lambda(\mathbf{x}_I) \subset \Lambda$ is the locally (p, q)-layered at $\mathbf{x}_I$, then the moment matrix at $\mathbf{x}_I$ can be calculated from the following manner:

(4.6)

$$M^{\rho_\mathbf{x}}(\mathbf{x}_I) = W(\mathbf{0})\,\mathbf{B}_m(\mathbf{0})\mathbf{B}_m(\mathbf{0})^T + \sum_{K=1}^{p} \delta_K\,\mathbf{D}_K \left(\sum_{\xi_J \in S_K} \mathbf{B}_m(\zeta_J)\mathbf{B}_m(\zeta_J)^T\right) \mathbf{D}_K,$$

where S_K is the Kth layer set in the definition and for any nonzero $\xi_J \in S_K$ we use the symbols

$$\xi_J \equiv T_{\mathbf{x}_I}(\mathbf{x}_J), \quad \tau_K \equiv |\xi_1| = \cdots = |\xi_q| < 1, \quad \zeta_J \equiv \frac{\xi_J}{\tau_K}, \tag{4.7}$$

$$\delta_K \equiv W(\xi_1) = \cdots = W(\xi_q), \tag{4.8}$$

and $\mathbf{D}_K$ is the diagonal matrix such that

$$\mathbf{D}_K \equiv \mathrm{Diag}(\tau_K^{|\alpha_1|}, \tau_K^{|\alpha_2|}, \ldots, \tau_K^{|\alpha_L|}). \tag{4.9}$$

Since we have assumed $n = 2$, we have, without loss of generality, the ζ_J's distributed evenly on the layer S_K and represented by

$$\zeta_j = \left(\cos\left(\theta_K + j\,\frac{2\pi}{q}\right), \sin\left(\theta_K + j\,\frac{2\pi}{q}\right)\right), \quad j = 0, 1, \ldots, q-1, \tag{4.10}$$

where θ_K is the angle of the starting node ζ_1 on S_K. If the distribution of nodes around $\mathbf{x}_I$ is assumed to be the locally (p,q)-layered at $\mathbf{x}_I$, then, from the trigonometric identities in Appendix II, the term $\sum_{\xi_J \in S_K} \mathbf{B}(\zeta_J)\mathbf{B}(\zeta_J)^T$ in (4.6) has the following forms for the cases when $q = 4$ and $q \geq 5$:

- $\sum_{\xi_J \in S_K} \mathbf{B}(\zeta_J)\mathbf{B}(\zeta_J)^T$ when $q = 4$,

$$4 \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{3}{8} + \frac{1}{8}\cos 4\theta_K & \frac{1}{8}\sin 4\theta_K & \frac{1}{8} - \frac{1}{8}\cos 4\theta_K \\ 0 & 0 & 0 & \frac{1}{8}\sin 4\theta_K & \frac{1}{8} - \frac{1}{8}\cos 4\theta_K & -\frac{1}{8}\sin 4\theta_K \\ \frac{1}{2} & 0 & 0 & \frac{1}{8} - \frac{1}{8}\cos 4\theta_K & -\frac{1}{8}\sin 4\theta_K & \frac{3}{8} + \frac{1}{8}\cos 4\theta_K \end{bmatrix}; \tag{4.11}$$

- $\sum_{\xi_J \in S_K} \mathbf{B}(\zeta_J)\mathbf{B}(\zeta_J)^T$ when $q \geq 5$

$$q \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{3}{8} & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{8} & 0 & \frac{3}{8} \end{bmatrix}. \tag{4.12}$$

Here we note that, if the number of nodes on the layer S_K is greater than or equal to 5, then the moment matrix does not depend on θ_K. This means that the (p,q)-layered node distributions for $q \geq 5$ makes the rotation invariant moment matrix.

The strong meshfree Laplacian operator Δ^ρ at $\mathbf{x}_I$ on the (p,q)-layered node set $\Lambda(\mathbf{x}_I)$ can be calculated from the equivalent form:

$$\begin{aligned} \sum_{\mathbf{x}_J \in \Lambda(\mathbf{x}_I)} u_h(\mathbf{x}_J)\,\psi_J^\Delta(\mathbf{x}_I) &= \mathbf{d}_\Delta\, M^{\rho_{\mathbf{x}}}(\mathbf{x}_I)^{-1}\,\mathbf{B}_m(\mathbf{0})\,W(\mathbf{0})\,u_h(\mathbf{0}) \\ &+ \mathbf{d}_\Delta\, M^{\rho_{\mathbf{x}}}(\mathbf{x}_I)^{-1} \sum_{K=1}^{p} \delta_K\,\mathbf{D}_K\,\left[\mathbf{B}_m\left(\zeta_1^K\right) \;\; \mathbf{B}_m\left(\zeta_2^K\right) \;\; \cdots \;\; \mathbf{B}_m\left(\zeta_q^K\right)\right]\,\mathbf{u}_h^K, \end{aligned} \tag{4.13}$$

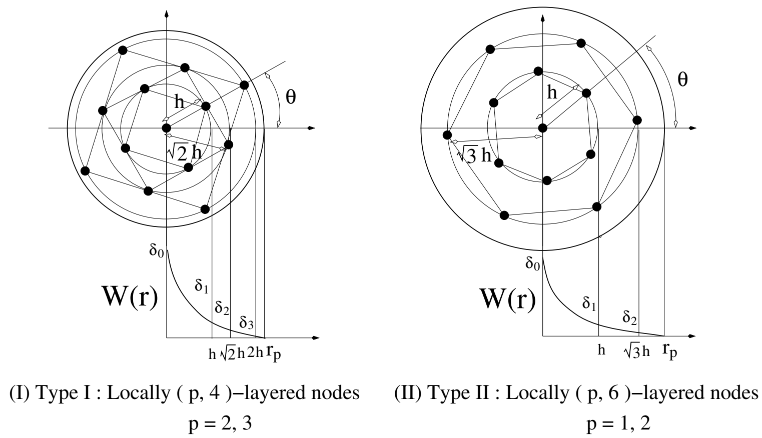

FIG. 4.2. *Two types of locally* (p, q)*-layered node distribution:* (I) *the set of Type* I *and,* (II) *the set of Type* II.

where $\mathbf{u}_h^K \equiv [u_h(\xi_1^K), \ldots, u_h(\xi_q^K)]^T$ is a column vector, the superscript of ξ_J^K means that ξ_J is on S_K, and when $m = 2$ the symbol $\mathbf{d}_\Delta$ designates the following row vector:

$$\mathbf{d}_\Delta \equiv \left[0,\ 0,\ 0,\ \frac{(2,0)!}{\rho_{\mathbf{x}_I}^{|(2,0)|}},\ 0,\ \frac{(0,2)!}{\rho_{\mathbf{x}_I}^{|(0,2)|}}\right]. \tag{4.14}$$

We now consider the two kinds of locally (p, q)-layered node distributions. The first one is composed of orthogonally positioned nodes and the other comes from a hexagonal structure.

In constructing specific types of nodes, we use the symbol δ_K as defined in (4.8) including $\delta_0 \equiv W(0) = 1$, or equivalently we have $\delta_K \equiv W(\tau_K)$ since our window function W in (2.5) depends only on the radial values. The terminology of the multi-index β_k defined in (3.3) is also utilized.

Let $h > 0$ and θ be the given angle.

4.1. Type I: The locally $(p, 4)$-layered nodes $(p = 2, 3)$. Let $A_{r_p}(\mathbf{0})$ be the set consisting of the following nodes as shown in Figure 4.2(I):

(4.15)

$$\{(0,0)\} \cup \bigcup_{K=1}^{p} \left\{ \left(t_K\, h\, \cos\left(\theta_K + i\,\frac{2\pi}{4}\right),\ t_K\, h\, \sin\left(\theta_K + i\,\frac{2\pi}{4}\right) \,\middle|\, i = 0,1,2,3 \right\},$$

where $t_K = \sqrt{2}^{K-1}$, $\theta_K = \theta + (K-1)\frac{\pi}{4}$, and

$$r_p = h\,\frac{\sqrt{2}+2}{2},\ h\,\frac{2+\sqrt{5}}{2},\ \text{respectively when } p = 2, 3. \tag{4.16}$$

If A is a subset of nodes with $\mathbf{x}_I$ as its center node and it has the same property as $A_{r_p}(\mathbf{0})$ for $p = 2, 3$ under the normalizing transform (4.5), then it is said to be *the set of Type I at the node* $\mathbf{x}_I$. In this case, the values of τ_K's in (4.7) are calculated as the following:

$$\tau_K = \frac{h\, t_K}{r_p}, \quad 1 \le K \le p = 2, 3. \tag{4.17}$$

When $p = 3$, the determinant of the moment matrix in (4.6) at the center node can be calculated such as

$$|M^{r_p}(\mathbf{0})| = 2^6 \tau_1^{2\sum_{k=1}^{6}|\beta_k|}\, \delta_2\, (\delta_1 + 2\,\delta_2 + 4\,\delta_3)^2\, (\delta_1 + 16\,\delta_3)\, A_R \neq 0, \tag{4.18}$$

where the symbol A_R stands for the following positive value:

$$A_R = \delta_1 + 4\,\delta_2 + 16\,\delta_3 + 4\,\delta_1\delta_2 + 16\,\delta_2\delta_3 + 36\,\delta_1\delta_3. \tag{4.19}$$

When $p = 2$, we may simply set $\delta_3 = 0$.

On this kind of local node distribution, the strong meshfree Laplacian operator at the origin is calculated from the equation (4.13) as follows:

$$\psi_{\mathbf{0}}^{\Delta}(\mathbf{0}) = -\frac{4}{h^2}\,\frac{\delta_0\,(\delta_1 + 2\,\delta_2 + 4\,\delta_3)}{A_R}, \tag{4.20}$$

$$\psi_{\xi_J \in S_1}^{\Delta}(\mathbf{0}) = \frac{1}{h^2}\,\frac{\delta_1\,(1 - 4\,\delta_2 - 12\,\delta_3)}{A_R} \equiv \frac{1}{h^2}\,A_1, \tag{4.21}$$

$$\psi_{\xi_J \in S_2}^{\Delta}(\mathbf{0}) = \frac{1}{h^2}\,\frac{2\,\delta_2\,(1 + 2\,\delta_1 - 4\,\delta_3)}{A_R} \equiv \frac{1}{h^2}\,A_2, \tag{4.22}$$

$$\psi_{\xi_J \in S_3}^{\Delta}(\mathbf{0}) = \frac{1}{h^2}\,\frac{4\,\delta_3\,(1 + 3\,\delta_1 + 2\,\delta_2)}{A_R} \equiv \frac{1}{h^2}\,A_3, \tag{4.23}$$

where S_K $(K = 1, 2, 3)$ is the Kth layer and there are 4-nodes on each S_K. Therefore, from (4.20), (4.21), (4.22), and (4.23) and the fact that $A_1 + A_2 + A_3 = \frac{\delta_1 + 2\,\delta_2 + 4\,\delta_3}{A_R}$ since $\delta_0 \equiv W(\mathbf{0}) = 1$, we have, for any $v \in V$,

$$h^2 \sum_{\mathbf{x}_J \in A_r(\mathbf{0})} v(\mathbf{x}_J)\,\psi_J^{\Delta}(\mathbf{0}) = \sum_{k=1}^{3} A_k \left(-4\,v(\mathbf{0}) + \sum_{\xi_J \in S_k} v(\xi_J)\right). \tag{4.24}$$

If we set $\delta_3 = 0$ in the above formula, then we obtain the case when $p = 2$. With these types of nodes, the moment matrix depends on the rotation of nodes (θ) but the discrete Laplacian shape function $\psi_J^{\Delta}(\mathbf{0})$ is invariant under the rotation.

Summarizing the above discussion, we have the following lemma on the set of Type I at $\mathbf{x}_I$.

LEMMA 1. *Let $(\Omega, \Lambda, \rho_{\mathbf{x}})$ be a proper triple and $\Lambda(\mathbf{x}_I) \subset \Lambda$ be the set of Type I at the node $\mathbf{x}_I$. Then we have the following properties:*

1. *The following representation formula for the strong meshfree Laplacian operator holds:*

$$\sum_{\mathbf{x}_J \in \Lambda(\mathbf{x}_I)} v(\mathbf{x}_J)\,\psi_J^{\Delta}(\mathbf{x}_I) = \frac{1}{h^2} \sum_{K=1}^{p} A_K \left(-4\,v(\mathbf{x}_I) + \sum_{\xi_J \in S_K} v(\xi_J)\right) \tag{4.25}$$

for some coefficients A_K which depend on the window function.

2. *If the coefficients A_K are positive, then the following type of inverse inequality for the Laplacian shape functions holds:*

$$\sum_{\mathbf{x}_J \in \Lambda(\mathbf{x}_I)} \left|\,\psi_J^{\Delta}(\mathbf{x}_I)\,\right| \leq \frac{8}{h^2}. \tag{4.26}$$

Proof. The first property directly comes from (4.24). For the second property, if $A_K > 0$ for $K = 1, \dots, p$ where $p = 2, 3$, then we have the following bound from (4.20), (4.21), (4.22), and (4.23):

$$\sum_{\mathbf{x}_J \in \Lambda(\mathbf{x}_I)} \left|\psi_J^{\Delta}(\mathbf{x}_I)\right| = -\psi_I^{\Delta}(\mathbf{x}_I) + \sum_{\mathbf{x}_J \in \Lambda(\mathbf{x}_I)\setminus\{\mathbf{x}_I\}} \psi_J^{\Delta}(\mathbf{x}_I) = \frac{8}{h^2}\sum_{K=1}^{p} A_K \le \frac{8}{h^2} \tag{4.27}$$

since $\psi_I^{\Delta}(\mathbf{x}_I)$ is the only nonpositive term among $\psi_J^{\Delta}(\mathbf{x}_I)$ for all J, $\mathbf{x}_J \in \Lambda(\mathbf{x}_I)$. ∎

4.2. Type II: The locally $(p, 6)$-layered nodes $(p = 1, 2)$. Let $A_{r_p}(\mathbf{0})$ be the set consisting of the following nodes as shown in Figure 4.2(II):

$$\{(0,0)\} \cup \bigcup_{K=1}^{p} \left\{ \left(t_K\, h \cos\left(\theta_K + i\,\frac{2\pi}{6}\right),\ t_K\, h \sin\left(\theta_K + i\,\frac{2\pi}{6}\right) \,\middle|\, i = 0,1,2,3,4,5 \right\}, \tag{4.28}$$

where $t_K = \sqrt{3}^{K-1}$, $\theta_K = \theta + (K-1)\frac{\pi}{6}$, and

$$r_p = h\frac{\sqrt{3}^{p-1} + \sqrt{3}^{p}}{2}, \quad p = 1, 2. \tag{4.29}$$

If A is a subset of nodes with $\mathbf{x}_I$ as its center node and it has the same property as $A_{r_p}(\mathbf{0})$ for $p = 1, 2$ under the normalizing transform (4.5), then it is said to be *the set of Type II at the node* $\mathbf{x}_I$. Here, we can see that

$$\tau_K = \frac{h\, t_K}{r_p}, \quad 1 \le K \le p = 1, 2. \tag{4.30}$$

When $p = 2$, the determinant of the moment matrix in (4.6) at the center node in this case can be calculated as follows:

$$|M^{r_p}(\mathbf{0})| = 3^5 \tau_1^{2\sum_{k=1}^{6}|\beta_k|} (\delta_1 + 3\,\delta_2)^2 (\delta_1 + 9\,\delta_2)^2 A_H \ne 0, \tag{4.31}$$

where the symbol A_H means the following positive value:

$$A_H = \delta_1 + 9\,\delta_2 + 24\,\delta_1\delta_2. \tag{4.32}$$

In the case when $p = 1$, we can set $\delta_2 = 0$.

On this kind of local node distribution, the strong meshfree Laplacian operator at the origin is derived from (4.13) as follows:

$$\psi_{\mathbf{0}}^{\Delta}(\mathbf{0}) = -\frac{4}{h^2}\,\frac{\delta_0\,(\delta_1 + 3\,\delta_2)}{A_H}, \tag{4.33}$$

$$\psi_{\xi_J \in S_1}^{\Delta}(\mathbf{0}) = \frac{1}{h^2}\,\frac{\frac{2}{3}\delta_1\,(1 - 12\,\delta_2)}{A_H} \equiv \frac{1}{h^2}A_1, \tag{4.34}$$

$$\psi_{\xi_J \in S_2}^{\Delta}(\mathbf{0}) = \frac{1}{h^2}\,\frac{2\delta_2\,(1 + 4\,\delta_1)}{A_H} \equiv \frac{1}{h^2}A_2, \tag{4.35}$$

where S_K $(K = 1, 2)$ is the Kth layer and there are 6-nodes in each S_K. Thus, from (4.33), (4.34), (4.35), and the identity $A_1 + A_2 = \frac{\frac{2}{3}(\delta_1 + 3\,\delta_2)}{A_H}$, we also obtain, for any $v \in V$,

$$h^2 \sum_{\mathbf{x}_J \in A_r(\mathbf{0})} v(\mathbf{x}_J)\,\psi_J^{\Delta}(\mathbf{0}) = \sum_{k=1}^{2} A_k \left(-6\,v(\mathbf{0}) + \sum_{\xi_J \in S_k} v(\xi_J) \right). \tag{4.36}$$

In the case when $p = 1$, we can only set $\delta_2 = 0$.

Summarizing the above discussion, on the set of Type II at $\mathbf{x}_I$, we have the following lemma similar to Lemma 1.

LEMMA 2. *Let $(\Omega, \Lambda, \rho_{\mathbf{x}})$ be a proper triple and $\Lambda(\mathbf{x}_I) \subset \Lambda$ be the set of Type II at the node $\mathbf{x}_I$. Then we have the following properties:*

1. *The following representation formula for the strong meshfree Laplacian operator holds:*

$$\sum_{\mathbf{x}_J \in \Lambda(\mathbf{x}_I)} v(\mathbf{x}_J)\, \psi_J^{\Delta}(\mathbf{x}_I) = \frac{1}{h^2} \sum_{K=1}^{p} A_K \left(-6\, v(\mathbf{x}_I) + \sum_{\xi_J \in S_K} v(\xi_J) \right) \tag{4.37}$$

for some coefficients A_K which depend on the window function.

2. *If the coefficients A_K are positive, then the following inverse inequality for the Laplacian shape functions holds:*

$$\sum_{\mathbf{x}_J \in \Lambda(\mathbf{x}_I)} \left| \psi_J^{\Delta}(\mathbf{x}_I) \right| \leq \frac{8}{h^2}. \tag{4.38}$$

Proof. The proof is similar to that in Lemma 1, so we omit the proof. ∎

Remark 3. The set of Type I and Type II belong to the locally $(p, 4)$-layered and locally $(p, 6)$-layered class, respectively. The significant feature of the set of Type I and Type II nodes is the staggered distribution of nodes across layers. Particularly, in the case of Type I, it is essential for the invertibility of the moment matrix at the center node since the matrix (4.11) derived from the nodes in each layer is singular with kernel dimension 1. However, in the case of Type II, the nodes do not have to be staggered through layers since one can see the nonsingular matrix (4.12) is independent of the attitude of nodes in each layer. Hence, Type II is more natural than Type I in the meshfree approximation.

4.3. Two possibly layered node distributions on specific domains. We propose two kinds of evenly spaced nodes on some domains. The size, the rotation, and the translation of the domain we are to construct are not critical in the subsequent analysis (i.e., the subsequent analysis is independent of the similarity transfromation).

As shown in Figure 4.3(a), we first consider the open square domain Ω_R with 4 vertices at

$$(1, 1),\ (-1, 1),\ (-1, -1),\ (1, -1). \tag{4.39}$$

In this case, the nodes can be distributed on $\overline{\Omega_R}$ to be Type I (i.e., staggered locally $(p, 4)$-layered $(p = 2, 3)$) at each interior node. The set of such nodes on Ω_R is written by the symbol Λ_R.

As depicted in Figure 4.3(b), the hexagonal domain Ω_H with the six vertices located at

$$(1, 0),\ \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right),\ \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),\ (-1, 0),\ \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),\ \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \tag{4.40}$$

is taken as the second open domain. On the domain $\overline{\Omega_H}$, the nodes can be entirely distributed to be Type II (i.e., locally $(p, 6)$-layered $(p = 1, 2)$) at each interior node. Such a set of nodes is denoted by Λ_H.

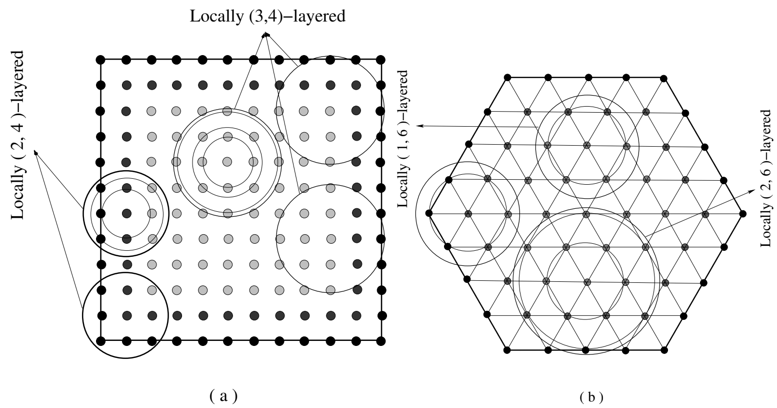

FIG. 4.3. *Possibly layered node distribution:* (a) Ω_R, *and* (b) Ω_H.

In both cases, the minimum node distance is set by $h \equiv \frac{2}{n}$ where n is regarded as the number of divisions. Now, we determine the dilation function $\rho_{\mathbf{x}}$ for each case. We only need the values of $\rho_{\mathbf{x}}$ at nodes. First of all, we notice that the value $\rho_{\mathbf{x}_I}$ at each node $\mathbf{x}_I$ on Λ_R (or Λ_H) depends on how we take the p number in the locally (p,q)-layered set $\Lambda(\mathbf{x}_I)$ at $\mathbf{x}_I$. As illustrated in Figure 4.3(a) and (b), we choose them in the following way:

$$(4.41)\quad \rho^R_{\mathbf{x}_I} = \begin{cases} h\,\frac{\sqrt{2}+2}{2}, & \mathbf{x}_I \notin \partial\Omega_R,\ \mathrm{dist}(\mathbf{x}_I, \partial\Omega_R) < \frac{3}{2}h \\ h\left((3-p)\frac{\sqrt{2}+2}{2} + (p-2)\frac{2+\sqrt{5}}{2}\right), & \mathbf{x}_I \notin \partial\Omega_R,\ \mathrm{dist}(\mathbf{x}_I, \partial\Omega_R) \geq \frac{3}{2}h \end{cases},$$

$$(4.42)\quad \rho^H_{\mathbf{x}_I} = \begin{cases} h\,\frac{1+\sqrt{3}}{2}, & \mathbf{x}_I \notin \partial\Omega_H,\ \mathrm{dist}(\mathbf{x}_I, \partial\Omega_H) < \frac{3}{2}h \\ h\left((2-p)\frac{1+\sqrt{3}}{2} + (p-1)\frac{\sqrt{3}+3}{2}\right), & \mathbf{x}_I \notin \partial\Omega_H,\ \mathrm{dist}(\mathbf{x}_I, \partial\Omega_H) \geq \frac{3}{2}h \end{cases}$$

in which $\mathrm{dist}(\mathbf{x}_I, B) \equiv \min_{\mathbf{y}\in B} \|\mathbf{x}_I - \mathbf{y}\|$ represents the distance between $\mathbf{x}_I$ and the closed set B as usual and $p = 2, 3$ and $p = 1, 2$, respectively, in (4.41) and (4.42). The dilation function values on the interior nodes, taken by (4.41) and (4.42), make the node sets Λ_R and Λ_H be the possibly layered. Furthermore, every $\Lambda_{\rho_{\mathbf{x}_I}}(\mathbf{x}_I)$ for interior node $\mathbf{x}_I$ becomes the set of Type I or Type II at $\mathbf{x}_I$.

On the boundary nodes for both cases, the dilation function values can be assigned arbitrarily but they must be large enough to ensure the inverse of the moment matrices at the nodes themselves. Actually, the dilation function values on the boundary nodes do not affect the subsequent theorems for the convergence proof.

From the construction of two triples, namely, $(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$ and $(\Omega_H, \Lambda_H, \rho^H_{\mathbf{x}})$, we can see that both Λ_R and Λ_H are the possibly layered and hence the two triples become the proper triples attributed to (4.24) and (4.36). Although we could not say how many possibly layered node distributions exist, we find at least two types of the possibly layered set of nodes.

4.4. Discrete maximum principle on the set of Type I or Type II. For these locally (p,q)-layered nodes of Type I and Type II, we should pay attention to the discretized form (4.24) and (4.36). If the window function is suitably chosen so that all the coefficients A_k may be strictly positive, then we can prove the discrete maximum principle at the center node.

LEMMA 3. *Let* $(\Omega, \Lambda, \rho_{\mathbf{x}})$ *be the triple. If the local node set* $\Lambda(\mathbf{x}_I) \subset \Lambda$ *is of either Type I or Type II at* $\mathbf{x}_I$*, then the strong meshfree Laplacian operator* Δ^ρ *satisfies the* *discrete maximum principle at the center node* $\mathbf{x}_I$.

Proof. Let $\mathbf{x}_I \in \Lambda$ be the center node of $\Lambda(\mathbf{x}_I)$ that is either Type I or Type II at $\mathbf{x}_I$. Then the set $\Lambda(\mathbf{x}_I)$ is obviously the locally (p, q)-layered where $p = 2, 3$ for $q = 4$ or $p = 1, 2$ for $q = 6$. The discrete Laplacian shape functions have been calculated in (4.24) and (4.36) for both cases. First, we will show that all the coefficients A_K of the representation formula in Lemmas 1 and 2 are strictly positive. In the case of Type I which is the $(p, 4)$-layered, we claim that, when $p = 3$,

$$1 - 4\,\delta_2 - 12\,\delta_3 > 0, \quad 1 + 2\,\delta_1 - 4\,\delta_3 > 0, \quad 1 + 3\,\delta_1 + 2\,\delta_2 > 0 \tag{4.43}$$

and, when $p = 2$,

$$1 - 4\,\delta_2 > 0, \quad 1 + 2\,\delta_1 > 0. \tag{4.44}$$

If $\delta_2 < \frac{1}{16}$ and $\delta_3 < \frac{1}{36}$, then all the left terms in (4.43) stay positive. Indeed, when $p = 3$, from (4.17) we have $\delta_2 = W(\frac{2\sqrt{2}}{2+\sqrt{5}}) < \frac{1}{16}$ and $\delta_3 = W(\frac{4}{2+\sqrt{5}}) < \frac{1}{36}$. In the other case when $p = 2$, it is true from (4.17) that $\delta_2 = W(\frac{2\sqrt{2}}{2+\sqrt{2}}) < \frac{1}{16}$. Therefore, we are done with the proof for the case of the locally $(p, 4)$-layered nodes $(p = 2, 3)$.

On the other hand, when $\Lambda(\mathbf{x}_I)$ is of Type II which is the locally $(p, 6)$-layered, all the coefficients A_1 and A_2 of the representation formula in Lemma 2 are positive since we know from (4.30) that $\delta_2 = W(\frac{2\sqrt{3}}{3+\sqrt{3}}) < \frac{1}{16}$ when $p = 2$. For the case when $\Lambda(\mathbf{x}_I)$ is the locally $(1, 6)$-layered at $\mathbf{x}_I$, we trivially have $A_1 > 0$.

Let $\Lambda(\mathbf{x}_I)$ be the set of Type I or Type II at $\mathbf{x}_I \in \Lambda$ as mentioned in this Lemma. To prove the discrete maximum principle, suppose $(\Delta^\rho\, v)(\mathbf{x}_I) \geq 0$ for some $v \in V$. Due to the window function of type (2.5), the coefficients A_i in Lemmas 1 or 2 are proved to be positive in the above. From the positivity of the coefficients of the representation formula in both Lemmas 1 and 2, it never happens under this assumption that the center nodal value v_I of v at $\mathbf{x}_I$ is strictly greater than all the other nodal values v_K at the node $\mathbf{x}_K \in \Lambda_{\rho_{\mathbf{x}_I}}(\mathbf{x}_I)$ and therefore we have

$$v_I \leq \max_{\mathbf{x}_K \in \Lambda(\mathbf{x}_I) \setminus \{\mathbf{x}_I\}} v_K. \tag{4.45}$$

If the equality holds, then the event $v_I > v_K$ for some $\mathbf{x}_K, K \neq I$ makes $(\Delta^\rho\, v)(\mathbf{x}_I) < 0$. Hence, all v_K's must be equal to v_I. Therefore, the operator Δ^ρ satisfies the discrete maximum principle at the node $\mathbf{x}_I$ and this completes the proof. ∎

4.5. A priori estimate for the strong meshfree Laplacian operator. For the set of nodes on which the discrete maximum principle holds, we can obtain the general results in the meshfree regime.

LEMMA 4. *Let $(\Omega, \Lambda, \rho_{\mathbf{x}})$ be a proper triple. Assume the operator Δ^ρ satisfies the discrete maximum principle on a finite subset $A \subset \Lambda$. Then we have the following inequality:*

$$\max_{\mathbf{x}_J \in A} v_J \leq \max_{\mathbf{x}_K \in A^* \setminus A} v_K \tag{4.46}$$

whenever $v \in V$ and $\Delta^\rho v \geq 0$ on A.

Proof. Let us assume $v \in V$ and $\Delta^\rho v \geq 0$ on A. Suppose the maximum of nodal values over A occurs at the node $\mathbf{x}_{K^*} \in A$; that is,

$$v_{K^*} = \max_{\mathbf{x}_J \in A} v_J. \tag{4.47}$$

Then only two cases are possible. The first case is when $\Lambda(\mathbf{x}_{K^*})\backslash A \neq \emptyset$. In this case, from the maximum principle we have nothing to prove. In the other case, we have $\Lambda(\mathbf{x}_{K^*}) \subset A$. For this case, all v_K's in $\Lambda(\mathbf{x}_{K^*})$ are the same as v_{K^*}. If we set $A_0 \equiv \Lambda(\mathbf{x}_{K^*})$, then we can construct the set $A_1 \equiv A_0^*$ strictly larger that A_0 (i.e., A_1 contains at least one node not in A_0). If $A_1\backslash A \neq \emptyset$, then this lemma is proved. If not, all coefficients v_K in A_1 must have the same value v_{K^*}. Continuing this process, we can construct $A_2 = A_1^*, A_3 = A_2^*, \dots$. However, this process has to stop in a number of finite steps since the number of nodes in A is finite. Therefore, we have proven this lemma. ∎

THEOREM 1 (a priori estimate for the strong meshfree Laplacian operator). *Let $(\Omega, \Lambda, \rho_{\mathbf{x}})$ be a proper triple. Assume the strong meshfree Laplacian operator Δ^ρ satisfies the discrete maximum principle on a finite subset $A \subset \Lambda$. Then, we have the following a priori estimate*

$$\|v\|_{\infty,A} \leq C(A)\, \|\Delta^\rho v\|_{\infty,A} + \|v\|_{\infty,A^*\backslash A} \quad \textit{whenever} \quad v \in V, \tag{4.48}$$

where $C(A) = \min_{\mathbf{x}_*} \max_{\mathbf{x}_L \in A^*\backslash A} \frac{1}{4}\|\mathbf{x}_L - \mathbf{x}_*\|^2$.

Proof. Let $v \in V$ be assumed to be the nodal function on Λ. Then from the definition of the strong meshfree Laplacian Δ^ρ, we have

$$(\Delta^\rho v)(\mathbf{x}_K) = \sum_{\mathbf{x}_J \in \Lambda} v_J\, \psi_J^\Delta(\mathbf{x}_K), \quad \mathbf{x}_K \in A. \tag{4.49}$$

Obviously we see that

$$-\|\Delta^\rho v\|_{\infty,A} \leq (\Delta^\rho v)(\mathbf{x}_K) \leq \|\Delta^\rho v\|_{\infty,A} \quad \text{for any} \quad \mathbf{x}_K \in A. \tag{4.50}$$

Owing to the reproducing property for polynomials up to second order, we have, for any $\mathbf{x}_K \in A$,

$$\Delta^\rho \left(\|\Delta^\rho v\|_{\infty,A}\, i \left(\frac{1}{4}\|\mathbf{x} - \mathbf{x}_*\|^2 \right) \right) = \sum_{\mathbf{x}_J \in \Lambda} \left(\|\Delta^\rho v\|_{\infty,A} \frac{1}{4}\|\mathbf{x}_J - \mathbf{x}_*\|^2 \right) \psi_J^\Delta(\mathbf{x}) \tag{4.51}$$

$$= \|\Delta^\rho v\|_{\infty,A}, \tag{4.52}$$

where $\mathbf{x}_*$ is an arbitrary point. The first equality (4.51) comes from the definition of the operator Δ^ρ on V. The last identity (4.52) enables us to derive the following inequalities due to (4.50). For any $\mathbf{x}_K \in A$,

$$\sum_{\mathbf{x}_J \in \Lambda} \left[v_J + \left(\|\Delta^\rho v\|_{\infty,A} \frac{1}{4}\|\mathbf{x}_J - \mathbf{x}_*\|^2 \right) \right] \psi_J^\Delta(\mathbf{x}_K) \geq 0, \tag{4.53}$$

$$\sum_{\mathbf{x}_J \in \Lambda} \left[-v_J + \left(\|\Delta^\rho v\|_{\infty,A} \frac{1}{4}\|\mathbf{x}_J - \mathbf{x}_*\|^2 \right) \right] \psi_J^\Delta(\mathbf{x}_K) \geq 0. \tag{4.54}$$

From the discrete maximum principle (4.46) in Lemma 4 and both inequalities (4.53) and (4.54), we can conclude that

$$v_K \leq \max_{\mathbf{x}_L \in A^*\backslash A} \left(v_L + \left(\|\Delta^\rho v\|_{\infty,A} \frac{1}{4}\|\mathbf{x}_L - \mathbf{x}_*\|^2 \right) \right), \tag{4.55}$$

$$-v_K \leq \max_{\mathbf{x}_L \in A^*\backslash A} \left(-v_L + \left(\|\Delta^\rho v\|_{\infty,A} \frac{1}{4}\|\mathbf{x}_L - \mathbf{x}_*\|^2 \right) \right) \tag{4.56}$$

for all $\mathbf{x}_K \in A$. Therefore, the following estimate holds:

$$|v_K| \le \max_{\mathbf{x}_L \in A^* \setminus A} |v_L| + \max_{\mathbf{x}_L \in A^* \setminus A} \left(\|\Delta^\rho v\|_{\infty,A} \frac{1}{4} \|\mathbf{x}_L - \mathbf{x}_*\|^2 \right). \tag{4.57}$$

This completes the proof. □

5. Error estimate for Poisson problem on specific domains Ω_R and Ω_H. From here, we will achieve the convergence of the numerical solutions using the meshfree point collocation approach (**DP**) in (3.1) for the Poisson equation with Dirichlet data on two specific domains—Ω_R and Ω_H. Through the convergence proof, we can also understand the basic phenomena on the strong meshfree Laplacian operator and can view the structure of the meshfree approximations.

For the numerical solution of the problem (**DP**), the meshfree point collocation scheme is proposed in the manner of (3.9). The existence and uniqueness of the numerical solutions of (**DP**) follows immediately from the a priori estimate in Theorem 1.

THEOREM 2 (existence and uniqueness). *Assume that* $(\Omega, \Lambda, \rho_{\mathbf{x}})$ *is either* $(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$ *or* $(\Omega_H, \Lambda_H, \rho^H_{\mathbf{x}})$. *Then there exists the unique solution of the problem* (**DP**) *on* V.

Proof. Let us introduce the linear mapping $\widehat{\Delta^\rho} : V \to V$ defined by

$$(\widehat{\Delta^\rho}\, v)(\mathbf{x}_K) \equiv \begin{cases} (\Delta^\rho\, v)(\mathbf{x}_K), & \mathbf{x}_K \in \Lambda^o \\ v_K, & \mathbf{x}_K \in \Lambda^b \end{cases} \quad \text{for all } v \in V, \tag{5.1}$$

where $\Lambda^o \equiv \Lambda \cap \Omega$ and $\Lambda^b \equiv \Lambda \setminus \Lambda^o$. Then our discrete problem (**DP**) is equivalent to the following:

$$\text{find } v \in V \text{ such that} \quad \widehat{\Delta^\rho}\, v = \begin{cases} i(f) & \text{on } \Lambda^o \\ g & \text{on } \Lambda^b \end{cases}. \tag{5.2}$$

Since Λ is the possibly layered, the discrete maximum principle holds on Λ^o. Applying the a priori estimate in Theorem 1 to the problem (5.2), we have

$$\|v\|_{\infty,\Lambda\cap\Omega} \le C\,\|i(f)\|_{\infty,\Lambda\cap\Omega} + \|g\|_{\infty,\Lambda\cap\partial\Omega}. \tag{5.3}$$

We claim that the linear mapping $\widehat{\Delta^\rho}$ is one-to-one and onto. It suffices to show that the mapping is one-to-one since solution space V has finite dimension.

Suppose $\widehat{\Delta^\rho}\, v = \mathbf{0}$. This means that f and g become zero on the right-hand side of (5.3). Consequently, we have $\|v\|_{\infty,\Lambda\cap\Omega} = 0$ and hence $v = 0$ on $\Lambda \cap \Omega$. This implies $v = 0$ on Λ since $g = 0$ on $\Lambda \cap \partial\Omega$. Therefore, the mapping $\widehat{\Delta^\rho}$ is injective. From the fact that $\operatorname{Im} \widehat{\Delta^\rho} = (\operatorname{Ker} \widehat{\Delta^\rho})^\perp = V$, we also are done with the surjective proof. □

Furthermore, the following error estimate of the unique nodal solution of the problem (**DP**) holds on two specific domains Ω_R and Ω_H under the regularity assumption of the continuous problem (**CP**).

THEOREM 3. *Let* $(\Omega, \Lambda, \rho_{\mathbf{x}})$ *be either the triple* $(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$ *or* $(\Omega_H, \Lambda_H, \rho^H_{\mathbf{x}})$. *Assume* $u \in C^0(\overline{\Omega}) \cap C^{s,\alpha}(\Omega)$ $(s = 3, 4)$ *is the classical solution of Poisson problem* (**CP**) *with Dirichlet data and* $u_h \in V$ *is the nodal solution of the discrete Poisson problem* (**DP**) *on the node set* Λ *corresponding to* Ω. *If* $\Lambda \cap \Omega$ *is the interior nodes of* Λ, *then we have the following error estimate:*

$$\|i(u) - u_h\|_{\infty,\Lambda\cap\Omega} \le K\, h^{s-2}\, \|u\|_{C^{s,\alpha}(\Omega)} \tag{5.4}$$

for some constant $K > 0$ independent of h.

Proof. We note that the set of nodes $\Lambda \equiv \Lambda_R$ (or Λ_H) of the proper triple $(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$ (or $(\Omega_H, \Lambda_H, \rho^H_{\mathbf{x}})$) is obviously the possibly layered from the construction. Thus we can calculate the operator Δ^ρ at every interior node $\mathbf{x}_I \in \Lambda \cap \Omega$. Let $\Lambda^o \equiv \Lambda \cap \Omega$ be the interior nodes of Λ. If $u_h \in V$ is the nodal solution of (**DP**) and $u \in C^0(\overline{\Omega}) \cap C^{s,\alpha}(\Omega)$ $(s = 3, 4)$ is the solution of (**CP**), then we can derive the error equation on Λ^o such that

$$(\Delta^\rho u_h)(\mathbf{x}_I) - \Delta u(\mathbf{x}_I) = 0 \quad \text{for all} \quad \mathbf{x}_I \in \Lambda^o. \tag{5.5}$$

From the error equation (5.5), we can obtain

$$\sum_{\mathbf{x}_J \in \Lambda} (u^h_J - u(\mathbf{x}_J))\, \psi^\Delta_J(\mathbf{x}_I) = \Delta u(\mathbf{x}_I) - \sum_{\mathbf{x}_J \in \Lambda} u(\mathbf{x}_J)\, \psi^\Delta_J(\mathbf{x}_I) \tag{5.6}$$

for all $\mathbf{x}_I \in \Lambda^o$. Since the domain Ω is convex, we can obtain the following Taylor expansions for $u(\mathbf{x})$ at $\mathbf{x}_I \in \Lambda^o$. For every $\mathbf{x}_J \in \Lambda(\mathbf{x}_I)$,

$$\begin{aligned} u(\mathbf{x}_J) = &\sum_{|\beta| \le s-1} \frac{1}{\beta!} (\mathbf{x}_J - \mathbf{x}_I)^\beta D^\beta u(\mathbf{x}_I) \\ &+ \sum_{|\beta| = s} \frac{1}{\beta!} \int_0^1 (1-\tau)^{s-1} D^\beta u(\mathbf{x}_I + \tau(\mathbf{x}_J - \mathbf{x}_I))\, d\tau (\mathbf{x}_J - \mathbf{x}_I)^\beta. \end{aligned} \tag{5.7}$$

Since $\Lambda(\mathbf{x}_I)$ is the locally (p,q)-layered $(q = 4, 6)$ at $\mathbf{x}_I \in \Lambda^o$, we can observe the symmetric node structure such that $-(\mathbf{x}_J - \mathbf{x}_I)$ and $\mathbf{x}_J - \mathbf{x}_I$ are on the same layer for all $\mathbf{x}_J \in \Lambda(\mathbf{x}_I)$. This implies that, when $|\beta| = 3$,

$$\sum_{\mathbf{x}_J \in \Lambda} (\mathbf{x}_J - \mathbf{x}_I)^\beta\, \psi^\Delta_J(\mathbf{x}_I) = 0. \tag{5.8}$$

Thus, inserting the expansions (5.7) into the right-hand side of (5.6), the following is obtained from the second order reproducing property and the symmetric factor (5.8):

$$\sum_{\mathbf{x}_J \in \Lambda} (u^h_J - u(\mathbf{x}_J))\, \psi^\Delta_J(\mathbf{x}_I) = \sum_{\mathbf{x}_J \in \Lambda} c_{IJ} \psi^\Delta_J(\mathbf{x}_I) \quad \text{for all} \quad \mathbf{x}_I \in \Lambda^o, \tag{5.9}$$

where the coefficients c_{IJ} are defined as

$$c_{IJ} = -\sum_{|\beta| = s} \frac{1}{\beta!} \int_0^1 (1-\tau)^{s-1} D^\beta u(\mathbf{x}_I + \tau(\mathbf{x}_J - \mathbf{x}_I))\, d\tau (\mathbf{x}_J - \mathbf{x}_I)^\beta. \tag{5.10}$$

Since the left-hand side of (5.9) is the image of the strong meshfree Laplacian operator Δ^ρ of $u^h - i(u) \in V$, the a priori estimate (4.48) due to the maximum principle on Λ^o leads to the following estimate

(5.11)
$$\max_{\mathbf{x}_J \in \Lambda^o} |u^h_J - u(\mathbf{x}_J)| \le C(\Lambda^o) \max_{\mathbf{x}_I \in \Lambda^o} \sum_{\mathbf{x}_J \in \Lambda(\mathbf{x}_I)} |c_{IJ}| \left|\psi^\Delta_J(\mathbf{x}_I)\right| + \max_{\mathbf{x}_J \in \Lambda^{o*} \setminus \Lambda^o} |u^h_J - u(\mathbf{x}_J)|.$$

In the case of the node distributions assumed, we know that $\Lambda^b = \Lambda^{o*} \setminus \Lambda^o$ and $u^h_J - u(\mathbf{x}_J) = 0$ on Λ^b because of the Dirichlet boundary conditions and hence the second term on the right-hand side of the inequality (5.11) vanishes. For the estimate of the first term on the right-hand side of the inequality (5.11), we need the estimate of $|c_{IJ}|$ for all $\mathbf{x}_J \in \Lambda(\mathbf{x}_I)$ as follows. For each $\mathbf{x}_I \in \Lambda^o$,

$$|c_{IJ}| \leq \left(\max_{|\beta|=s} \sup_{\mathbf{x}\in\Omega} |D^\beta u(x)| \right) \left(\max_{\mathbf{x}_J \in \Lambda(\mathbf{x}_I)} |\mathbf{x}_J - \mathbf{x}_I| \right)^s \frac{1}{s} \sum_{|\beta|=s} \frac{1}{\beta!} \tag{5.12}$$

$$\leq K_s \, \|u\|_{C^{s,\,\alpha}} \, \|i(\rho)\|^s_{\infty,\Lambda(\mathbf{x}_I)}, \tag{5.13}$$

where $K_s = \frac{1}{s}\sum_{|\beta|=s}\frac{1}{\beta!}$ and ρ is the dilation function. Therefore, we have the following error bound

$$\max_{\mathbf{x}_J\in\Lambda^o} |u_J^h - u(\mathbf{x}_J)| \leq K_s \, C(\Lambda^o) \, \|i(\rho)\|^s_{\infty,\Lambda^o} \, \|u\|_{C^{s,\,\alpha}} \max_{\mathbf{x}_I\in\Lambda^o} \sum_{\mathbf{x}_J\in\Lambda(\mathbf{x}_I)} \left|\psi_J^\Delta(\mathbf{x}_I)\right|. \tag{5.14}$$

On the other hand, the constant $C(\Lambda^o)$ is bounded by the diameter of the domain Ω, and the dilation function ρ in the assumed triple $(\Omega, \Lambda, \rho_{\mathbf{x}})$ satisfies

$$h < \|i(\rho)\|_{\infty,\Lambda^o} < C\,h \tag{5.15}$$

for some constant C independent of h. Furthermore, from Lemmas 1 and 2,

$$\sum_{\mathbf{x}_J\in\Lambda(\mathbf{x}_I)} \left|\psi_J^\Delta(\mathbf{x}_I)\right| \leq \frac{8}{h^2}. \tag{5.16}$$

Consequently, we obtain the error estimate derived from (5.14):

$$\|i(u) - u_h\|_{\infty,\Lambda\cap\Omega} \leq K\,h^{s-2}\,\|u\|_{C^{s,\alpha}(\Omega)} \tag{5.17}$$

for some $K > 0$ independent of h. □

Remark 4. As seen in the proof of Theorem 3, the convergence order of the numerical solution to the exact one can be proven only to be 2, although the regularity index s of the solution becomes greater than 4. The higher order of basis polynomials in the fast version of the generalized moving least square meshfree approximation is directly related to a lift in the convergence order (see [8]). Its proof seems to need the boundary error estimate for the numerical solution without the discrete maximum principle, while the interior error estimate is the same as ours.

The error ratio of about 4 in Table 5.1 implies the second order convergence even for the less-regularity case. The numerical result not only attests the validation of the error estimate but also shows the numerical scheme proposed could be more accurate than we anticipated. A numerical example is proposed to verify the theoretical convergence result. The solution $u(x, y)$ is assumed to be defined on both domains $(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$ and $(\Omega_H, \Lambda_H, \rho^H_{\mathbf{x}})$ as follows:

$$u(x, y) = e^{x+y-1} \left| x - \frac{1}{2} \right| \left(x - \frac{1}{2} \right)^2. \tag{5.18}$$

Applying the Laplacian operator to this solution, the corresponding force is given as follows:

$$f(x, y) = 2 \left| x - \frac{1}{2} \right| e^{x+y-1} \left(x^2 + 2x + \frac{7}{4} \right). \tag{5.19}$$

TABLE 5.1

Numerically experimental result on the relative error ($\|u^h - i(u)\|_{\Lambda,\infty}/\|i(u)\|_{\Lambda,\infty}$) *and the convergence rate of the numerical solutions for* $(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$ *and* $(\Omega_H, \Lambda_H, \rho^H_{\mathbf{x}})$*, where* Λ *is either* Λ_R *or* Λ_H*, and* ρ^R *and* ρ_H *are taken as the value* $1.8 * h$ *at each interior node so that* Λ_R *and* Λ_H *can be locally* $(2,4)$*-layered and locally* $(1,6)$*-layered, respectively.*

h	$(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$	Error ratio	h	$(\Omega_H, \Lambda_H, \rho^H_{\mathbf{x}})$	Error ratio
$\frac{2}{20}$	4.2660×10^{-3}	—	$\frac{2}{10}$	1.1900×10^{-2}	—
$\frac{2}{40}$	1.0726×10^{-3}	3.98	$\frac{2}{20}$	2.9148×10^{-3}	4.08
$\frac{2}{80}$	2.6857×10^{-4}	4.00	$\frac{2}{40}$	7.1881×10^{-4}	4.06
$\frac{2}{160}$	6.7162×10^{-5}	4.00	$\frac{2}{80}$	1.7833×10^{-4}	4.03

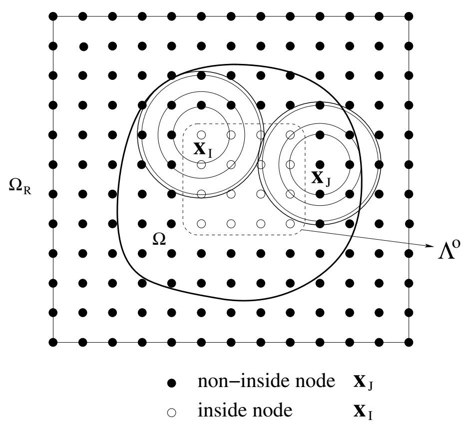

FIG. 5.1. *The immersed domain* Ω *in* Ω_R.

In this case, Dirichlet boundary condition on either $\partial\Omega_R$ or $\partial\Omega_H$ is presumed from the exact solution $u(x, y)$. The function u, in fact, belongs to $C^{2,1}$-class of functions on considered domains whose regularity is weaker than that stated in Theorem 3; nevertheless, the numerical example produces the second order convergence result as seen in Table 5.1.

6. Error estimate in the general domain immersed in Ω_R or Ω_H. We will try to analyze the convergence of our discrete problem (**DP**) with the boundary condition zero on a domain Ω which is immersed in the larger domain, for example, $\Omega \subset \hat{\Omega}_R$ (or $\Omega \subset \hat{\Omega}_H$) where $\hat{\Omega}_R$ (or $\hat{\Omega}_H$) is the image domain transformed from Ω_R (or Ω_H) by the similarity map. For brevity, we will rename it by Ω_R (or Ω_H).

Let $\Lambda \equiv \Lambda_R$ be the set of nodes in the triple $(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$. Assume that the nodal solution space V in this case is defined on Λ. As shown in Figure 5.1, we separate the set of nodes into two parts—the set Λ^o of inside nodes and the set Λ^b of non-inside nodes which are defined as the following:

$$\Lambda^o \equiv \{\mathbf{x}_J \in \Lambda \,|\, \Lambda(\mathbf{x}_J) \subset \Omega\}, \quad \Lambda^b \equiv \Lambda \setminus \Lambda^o. \tag{6.1}$$

The inside node $\mathbf{x}_I \in \Lambda$ implies that the $\rho_{\mathbf{x}_I}$-neighbor nodes are contained in the open set Ω while the nonside node is anything else. The following is the immersed meshfree

Poisson problem:

$$(6.2)\qquad (\mathbf{IMP})\begin{cases} u_h \in V_0 \equiv \{v_J \in \mathbb{R} \,|\, v_K = 0 \text{ for all } \mathbf{x}_K \in \Lambda^b\} \subset V \\ \Delta^\rho u_h = i(f) \quad \text{on } \Lambda^o \end{cases}.$$

Let the solution u of the Poisson problem (**CP**) with zero Dirichlet data belong to $C^0(\overline{\Omega})\cap C^{3,\,\alpha}(\Omega)$ and $u_h \in V_0$ be the nodal solution of the immersed meshfree Poisson problem (**IMP**). If we extend u to $\Omega_R \setminus \Omega$ by zero, then we conjecture that

$$(6.3)\qquad \|i(u) - u_h\|_{\infty,\Lambda\cap\Omega} \le K\,h\,\|u\|_{C^{3,\alpha}(\Omega)},$$

where the constant K is independent of h.

Theorem 4 (existence and uniqueness). *Let $(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$ and $(\Omega_H, \Lambda_H, \rho^H_{\mathbf{x}})$ be the triples. Assume Ω is immersed in either Ω_R or Ω_H. Then there exists the unique solution of the problem* (**IMP**) *on V.*

The proof of the theorem is similar to that of Theorem 2 and thus it is omitted.

In order to prove the convergence result (6.3) for the immersed meshfree Poisson problem (**IMP**), let $(\Omega_R, \Lambda_R, \rho^R_{\mathbf{x}})$ be the triple defined in section 5. First, we have the following error:

$$(6.4)\qquad \|i(u) - u_h\|_{\infty,\Lambda^{o\,*}\setminus\Lambda^o} = \|i(u)\|_{\infty,\Lambda^{o\,*}\setminus\Lambda^o}$$

since $\Lambda^{o\,*} \setminus \Lambda^o \subset \Omega$ and $u_h(\mathbf{x}_J) = 0$ on it. Through the similar procedure to the proof of Theorem 3 and from the fact that $\Lambda(\mathbf{x}_I) \subset \Omega$ for any $\mathbf{x}_I \in \Lambda^o$, we can obtain the following error equation:

$$(6.5)\qquad \sum_{\mathbf{x}_J\in\Lambda} (u^h_J - u(\mathbf{x}_J))\,\psi^\Delta_J(\mathbf{x}_I) = \sum_{\mathbf{x}_J\in\Lambda} c_{IJ}\psi^\Delta_J(\mathbf{x}_I) \quad \text{for all} \quad \mathbf{x}_I \in \Lambda^o,$$

where the coefficients c_{IJ} are calculated as follows:

$$(6.6)\qquad c_{IJ} = -\sum_{|\beta|=3} \frac{1}{\beta!}\int_0^1 (1-\tau)^2 D^\beta u(\mathbf{x}_I + \tau(\mathbf{x}_J - \mathbf{x}_I))\,d\tau(\mathbf{x}_J - \mathbf{x}_I)^\beta.$$

From a priori estimate in Theorem 1 due to the discrete maximum principle on Λ^o and from the identity (6.4), we obtain the following estimates:

$$\begin{aligned} \|i(u) - u_h\|_{\infty,\Lambda^o} &\le C(\Lambda^o)\max_{\mathbf{x}_I\in\Lambda^o}\left|\sum_{\mathbf{x}_J\in\Lambda} c_{IJ}\psi^\Delta_J(\mathbf{x}_I)\right| + \|i(u) - u_h\|_{\infty,\Lambda^{o\,*}\setminus\Lambda^o} \\ (6.7)\qquad &\le K_1\,h\,\|u\|_{C^{3,\alpha}(\Omega)} + \|i(u)\|_{\infty,\Lambda^{o\,*}\setminus\Lambda^o} \end{aligned}$$

for some constant $K_1 > 0$ independent of h. On the other hand, let us pay attention to the fact that

$$(6.8)\qquad \|i(u) - u_h\|_{\infty,\Lambda^b\cap\Omega} = \|i(u)\|_{\infty,\Lambda^b\cap\Omega}.$$

Then, from this fact and (6.7), the nodal error on the nodes in Ω is bounded by

$$(6.9)\qquad \|i(u) - u_h\|_{\infty,\Lambda\cap\Omega} \le K_1\,h\,\|u\|_{C^{3,\alpha}(\Omega)} + \|i(u)\|_{\infty,\Lambda^b\cap\Omega}$$

since $\Lambda^{o*} \setminus \Lambda^o \subset \Lambda^b \cap \Omega$. Here, the second term on the right-hand side of (6.9) is bounded by

$$\|i(u)\|_{\infty,\Lambda^b\cap\Omega} \le \left(\max_{\mathbf{x}_K\in\Lambda^b\cap\Omega} \mathrm{dist}(\mathbf{x}_K,\partial\Omega)\right) \|u\|_{C^{1,\alpha}(\Omega)} \le K_2\, h\, \|u\|_{C^{3,\alpha}(\Omega)} \tag{6.10}$$

for some constant $K_2 > 0$ independent of h. Therefore, we have obtained the following theorem.

THEOREM 5. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain which is immersed in either Ω_R or Ω_H. Assume that $u \in C^0(\overline{\Omega}) \cap C^{3,\alpha}(\Omega)$ is the solution of* (**CP**) *and $u_h \in V_0$ is the nodal solution of* (**IMP**). *Then we have the following error estimate:*

$$\|i(u) - u_h\|_{\infty,\Lambda\cap\Omega} \le K\, h\, \|u\|_{C^{3,\alpha}(\Omega)}, \tag{6.11}$$

where the node set Λ is either Λ_R or Λ_H and K is constant independent of h.

7. Conclusion. The generalized moving least square approximation is introduced, and based on this, we can define the strong meshfree Laplacian operator in the sense of a point collocation strategy. From the mathematical point of view, the discrete maximum principle for the strong meshfree Laplacian operator is presented for several types of layered node distributions. Using this principle, we perform convergence analysis for the nodal solutions of the Poisson problem with Dirichlet data on the boundary. As a result, second order convergence is achieved on the specific nodes in two typical domains, while the generally shaped domain immersed in these domains produces first order convergence of the nodal solution. An a priori estimate for the strong Laplacian operator in the meshfree regime is newly obtained via the discrete maximum principle and it is located in the core of the convergence proof together with the point collocation scheme proposed in this paper.

Appendix I: Generalized moving least square reproducing operators. For a given window function $W(\mathbf{x})$ and a dilation function $\rho_{\mathbf{x}}$, we find the vector $\mathbf{a}$ to minimize the following weighted square functional at $\bar{\mathbf{x}} \in \overline{\Omega}$:

$$J(\mathbf{a};\bar{\mathbf{x}},u) \equiv \sum_{\mathbf{x}_I\in\Lambda} |u(\mathbf{x}_I) - \mathbf{U}_m^{\rho_{\bar{\mathbf{x}}}}(\mathbf{x}_I;\bar{\mathbf{x}},\mathbf{a})|^2\, W\left(\frac{\mathbf{x}_I - \bar{\mathbf{x}}}{\rho_{\bar{\mathbf{x}}}}\right), \tag{7.1}$$

where $u(\mathbf{x})$ is a continuous function defined in $\overline{\Omega}$ and $\mathbf{U}_m^{\rho_{\bar{\mathbf{x}}}}(\mathbf{x};\bar{\mathbf{x}},\mathbf{a}) \equiv \mathbf{B}_m(\frac{\mathbf{x}-\bar{\mathbf{x}}}{\rho_{\bar{\mathbf{x}}}}) \cdot \mathbf{a}$. Then the minimizer $\mathbf{a}$ should be a function of $\bar{\mathbf{x}}$ and u, and we can make the following approximation operators for u by limiting process

$$D^{\beta_k}_{m,\rho_{\mathbf{x}}}\, u(\mathbf{x}) \equiv \lim_{\bar{\mathbf{x}}\to\mathbf{x}} D^{\beta_k}_{\mathbf{x}} \mathbf{U}_m^{\rho_{\bar{\mathbf{x}}}}(\mathbf{x};\bar{\mathbf{x}},\mathbf{a}(\bar{\mathbf{x}},u)), \quad |\beta_k| \le m. \tag{7.2}$$

As a matter of fact, the operators $D^{\beta_k}_{m,\rho_{\mathbf{x}}}$ are linear in $u(\mathbf{x})$. We call the operator $D^{\beta}_{m,\rho_{\mathbf{x}}}$ ($|\beta| \le m$) the βth meshfree approximated derivative operator equipped with $\rho_{\mathbf{x}}$.

Suppose $\{u_I(\mathbf{x}) \,|\, u_I(\mathbf{x}_J) = \delta_{IJ},\ \mathbf{x}_I, \mathbf{x}_J \in \Lambda\}$ is a set of continuous functions. We define the following functions:

$$\psi_I^{\rho_{\mathbf{x}},[\beta_k]}(\mathbf{x}) \equiv D^{\beta_k}_{m,\rho_{\mathbf{x}}}\, u_I(\mathbf{x}). \tag{7.3}$$

Then the functions $\psi_I^{\rho_{\mathbf{x}},[\beta_k]}(\mathbf{x})$ can be characterized as follows:

$$\begin{pmatrix} {\rho_{\mathbf{x}}}^{|\beta_1|}\,\psi_I^{\rho_{\mathbf{x}},[\beta_1]}(\mathbf{x}) \\ {\rho_{\mathbf{x}}}^{|\beta_2|}\,\psi_I^{\rho_{\mathbf{x}},[\beta_2]}(\mathbf{x}) \\ \vdots \\ {\rho_{\mathbf{x}}}^{|\beta_L|}\,\psi_I^{\rho_{\mathbf{x}},[\beta_L]}(\mathbf{x}) \end{pmatrix} = J_{\mathbf{B}_m}(\mathbf{0})\, M^{\rho_{\mathbf{x}}}(\mathbf{x})^{-1}\,\mathbf{B}_m\left(\frac{\mathbf{x}_I-\mathbf{x}}{\rho_{\mathbf{x}}}\right)\, W\left(\frac{\mathbf{x}_I-\mathbf{x}}{\rho_{\mathbf{x}}}\right), \tag{7.4}$$

where $M^{\rho_{\mathbf{x}}}(\mathbf{x})$ is called the moment matrix and is defined such that

$$M^{\rho_{\mathbf{x}}}(\mathbf{x}) \equiv \sum_{\mathbf{x}_I\in\Lambda} \mathbf{B}_m\left(\frac{\mathbf{x}_I-\mathbf{x}}{\rho_{\mathbf{x}}}\right)\mathbf{B}_m^T\left(\frac{\mathbf{x}_I-\mathbf{x}}{\rho_{\mathbf{x}}}\right) W\left(\frac{\mathbf{x}_I-\mathbf{x}}{\rho_{\mathbf{x}}}\right). \tag{7.5}$$

We call the function $\psi_I^{\rho_{\mathbf{x}},[\beta]}(\mathbf{x})$ *the βth shape function* associated with the window function W and the dilation function $\rho_{\mathbf{x}}$, or briefly call it *the βth shape function* if no confusion arises. As a consequence of (7.3), the operator $D_{m,\rho_{\mathbf{x}}}^{\beta_k}$ defined by (7.2) can be rewritten as follows:

$$D_{m,\rho_{\mathbf{x}}}^{\beta} u(\mathbf{x}) = \sum_{\mathbf{x}_J\in\Lambda} u(\mathbf{x}_J)\,\psi_J^{\rho_{\mathbf{x}},[\beta]}(\mathbf{x}), \quad |\beta| \le m. \tag{7.6}$$

We also call this operator the βth meshfree approximated derivative operator and the following properties of this operator can be justified.

THEOREM 6 (generalized mth order consistency). *We have the following identities:*

$$\sum_{\mathbf{x}_I\in\Lambda} b_\alpha\left(\frac{\mathbf{x}_I-\mathbf{x}}{\rho_{\mathbf{x}}}\right)\,\psi_I^{\rho_{\mathbf{x}},[\beta]}(\mathbf{x}) = \frac{1}{{\rho_{\mathbf{x}}}^{|\beta|}}\,\frac{\partial^\beta}{\partial \mathbf{x}^\beta} b_\alpha(\mathbf{0}). \tag{7.7}$$

Proof. To the matrix equation (7.4) for *the βth shape functions*, multiplying $\mathbf{B}_m(\mathbf{x}_I - \mathbf{x}/\rho_{\mathbf{x}})$ to the right on both sides and summing it over the whole nodes $\mathbf{x}_I$, we obtain the matrix equation. If we rewrite it in element-wise manner, then we have the resultant (7.7). □

The above theorem does not promise the βth meshfree approximated derivative operator to reproduce automatically all of the derivatives for the basis functions. However, for some useful class of functions including the polynomial class up to order m, we can have the generalized reproducing property which will play an important role in the convergence of approximations. For the class of the given basis functions to have such a generalized reproducing property, it is sufficient to satisfy the following condition.

COROLLARY 1 (sufficient condition for the generalized reproducing property). *Under the constant dilation function such that $\rho_{\mathbf{x}} \equiv \rho$, we assume that the basis functions satisfy the following relationships:*

$$b_\beta\left(\frac{\mathbf{y}}{\rho}\right) = \sum_{|\gamma|\le m} c_{\gamma\,\beta}\left(\frac{\mathbf{x}}{\rho}\right) b_\gamma\left(\frac{\mathbf{y}-\mathbf{x}}{\rho}\right), \quad |\beta| \le m \tag{7.8}$$

and the coefficient matrix is calculated from the equation

$$\left[c_{\alpha\,\beta}\left(\frac{\mathbf{x}}{\rho}\right)\right] \equiv C\left(\frac{\mathbf{x}}{\rho}\right) = J_{\mathbf{B}_m}(0)^{-1}\, J_{\mathbf{B}_m}\left(\frac{\mathbf{x}}{\rho}\right), \tag{7.9}$$

where $J_{\mathbf{B}_m}\left(\frac{\mathbf{x}}{\rho}\right)$ *is the Jacobian matrix defined as*

$$J_{\mathbf{B}_m}\left(\frac{\mathbf{x}}{\rho}\right) \equiv \left[(D^\alpha b_\beta)\left(\frac{\mathbf{x}}{\rho}\right)\right]. \tag{7.10}$$

Then the basis functions scaled by ρ *are exactly reproduced by the meshfree approximated derivative operators, i.e.,*

$$D^\beta_{m,\rho_\mathbf{x}}\, b_\beta\left(\frac{\mathbf{x}}{\rho}\right) = D^\beta_\mathbf{x}\, b_\beta\left(\frac{\mathbf{x}}{\rho}\right), \quad |\beta| \le m. \tag{7.11}$$

Proof. Assume that $b_\alpha(\mathbf{x})$'s for $|\alpha| \le m$ are the basis functions satisfying both conditions of (7.8) and (7.9). If we directly enforce (7.7) on these basis functions, then we obtain the result of (7.11). □

Corollary 1 provides us the opportunity of taking the general basis functions which can be reproduced in a dilated form. It is worth noting that the reproducing property does not happen in general if we take an arbitrary set of basis functions. That is why we propose the sufficient condition to ensure the reproducing condition for the dilated basis functions. According to the sufficient condition of (7.8) and (7.9) for the reproducing of basis functions, the class of polynomial basis up to order m can be shown to satisfy the exact reproducing property. That is, all of the derivatives of the basis itself are reproducible even in the case when involving the dilation function.

Corollary 2. *If we take the polynomials up to order* m *as basis functions, then the* β*th meshfree approximated derivative operator* $D^\beta_{m,\rho_\mathbf{x}}$ *is exactly the same as the differential operator* D^β *on the polynomial space up to order* m*. That is,*

$$D^\beta_{m,\rho_\mathbf{x}}\, u(\mathbf{x}) = D^\beta_\mathbf{x}\, u(\mathbf{x}) \tag{7.12}$$

whenever $u(\mathbf{x})$ *is a polynomial of order up to* m*.*

Proof. We can replace all $\rho_\mathbf{x}$ in Theorem 6 and all ρ in Corollary 1 with the number 1 for the case of polynomial basis up to order m. This fact suffices to prove this lemma. □

This corollary can be understood by recognizing that the βth meshfree approximated derivative operator $D^\beta_{m,\rho_\mathbf{x}}$ behaves in the same way as the exact derivative operator $D^\beta_\mathbf{x}$ at least on the polynomial function space up to order m.

Appendix II: Trigonometric identities. Let θ be an angle fixed. Then we have the following trigonometric identities for any natural number $n \geq 4$:

(7.13)
$$\sum_{k=0}^{n-1} \cos\left(\theta + k\,\frac{2\pi}{n}\right) = \sum_{k=0}^{n-1} \sin\left(\theta + k\,\frac{2\pi}{n}\right) = \sum_{k=0}^{n-1} \cos\left(\theta + k\,\frac{2\pi}{n}\right)\,\sin\left(\theta + k\,\frac{2\pi}{n}\right) = 0,$$

(7.14)
$$\sum_{k=0}^{n-1} \cos^2\left(\theta + k\,\frac{2\pi}{n}\right) = \sum_{k=0}^{n-1} \sin^2\left(\theta + k\,\frac{2\pi}{n}\right) = \frac{n}{2},$$

(7.15)
$$\sum_{k=0}^{n-1} \cos^3\left(\theta + k\,\frac{2\pi}{n}\right) = \sum_{k=0}^{n-1} \sin^3\left(\theta + k\,\frac{2\pi}{n}\right) = 0,$$

(7.16)
$$\sum_{k=0}^{n-1} \cos^2\left(\theta + k\,\frac{2\pi}{n}\right)\,\sin\left(\theta + k\,\frac{2\pi}{n}\right) = \sum_{k=0}^{n-1} \cos\left(\theta + k\,\frac{2\pi}{n}\right)\,\sin^2\left(\theta + k\,\frac{2\pi}{n}\right) = 0,$$

(7.17)
$$\sum_{k=0}^{n-1} \cos^4\left(\theta + k\,\frac{2\pi}{n}\right) = \sum_{k=0}^{n-1} \sin^4\left(\theta + k\,\frac{2\pi}{n}\right) = \begin{cases} \frac{3}{8}n + \frac{1}{8}n\,\cos 4\theta, & n = 4 \\ \frac{3}{8}n, & n \neq 4 \end{cases},$$

(7.18)
$$\sum_{k=0}^{n-1} \cos^2\left(\theta + k\,\frac{2\pi}{n}\right)\,\sin^2\left(\theta + k\,\frac{2\pi}{n}\right) = \begin{cases} \frac{1}{8}n - \frac{1}{8}n\,\cos 4\theta, & n = 4 \\ \frac{1}{8}n, & n \neq 4 \end{cases},$$

$$\sum_{k=0}^{n-1} \cos^3\left(\theta + k\,\frac{2\pi}{n}\right)\,\sin\left(\theta + k\,\frac{2\pi}{n}\right) = \begin{cases} \frac{1}{8}n\,\sin 4\theta, & n = 4 \\ 0, & n \neq 4 \end{cases}, \tag{7.19}$$

$$\sum_{k=0}^{n-1} \cos\left(\theta + k\,\frac{2\pi}{n}\right)\,\sin^3\left(\theta + k\,\frac{2\pi}{n}\right) = \begin{cases} -\frac{1}{8}n\,\sin 4\theta, & n = 4 \\ 0, & n \neq 4 \end{cases}. \tag{7.20}$$

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