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MESHFREE METHOD FOR THE NON-STATIONARY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We consider the solvability and the error estimates of numerical solutions of the non-stationary incompressible Stokes and Navier-Stokes equations by the meshfree method. The moving least square reproducing kernel method or the MLSRK method is employed for the space approximations. The existence of numerical solutions and the L^2-type error estimates are obtained. As a numerical example, we compare the numerical solutions of the Stokes and the Navier-Stokes equations with the exact solutions. Also we solve the non-stationary Navier-Stokes driven cavity flow using the MLSRK method.

1. **Introduction.** In this paper, the solvability and the error estimates of numerical solutions for the non-stationary incompressible Stokes and Navier-Stokes equations are considered by a Galerkin formulated meshfree method.

Several meshfree methods were proposed to various applications. We note that Smoothed Particle Hydrodynamics (SPH) by Gingold and Monaghan(1977) [4], Reproducing Kernel Particle Method (RKPM) by Liu et al.(1995) [12] [13], Diffuse Element Method (DEM) by Nayrole et al.(1992) [17], Element Free Galerkin Method (EFG) by Belytschko et al.(1994) [15], Partition of Unity Finite Element Method (PUFEM) by Babuška and Melenk(1995) [16], Meshfree point collocation method (MPCM) by Aluru(2000) [1] and Fast Moving Least Square Reproducing Kernel Method (FMLSRK) by Kim and Kim(2003) [7] were proposed. In particular, we are interested in the applications of the Moving Least Square Reproducing Kernel Galerkin Method (MLSRK) proposed by Liu et al.(1996) [14] to the non-stationary incompressible Navier-Stokes equations.

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Comparing the extensive development of methodology, the mathematical study on the meshfree method is under progress. In the paper[3], we obtained the solvability and the convergence of numerical solutions for the stationary incompressible Stokes and Navier-Stokes equations. As far as we know, this was the first attempt to address mathematical result for meshfree method to the incompressible Stokes and Navier-Stokes equations. In this paper, we consider time dependent incompressible flow problems and developed a somewhat independent theory compared to the stationary case. The approximation scheme for the time is not specified in this paper. Our interest is the solvability and the error estimates for time continuous and space discrete Stokes and Navier-Stokes equations. We obtain the solvability and the convergence for successive approximation for the solution of the non-stationary Stokes and Navier-Stokes equations, which results in the L^2-error estimate of the velocity.

For verification of the theory, we calculate numerical solutions for the non-stationary incompressible Stokes and Navier-Stokes equations in two dimension and relative errors of numerical solution are tabulated. Also the driven cavity flow is implemented numerically as a test problem.

2. **Moving Least Square Reproducing Kernel Method.** In this section, we only describe the outline of the MLSRK approximation. Let Ω be a bounded domain in $\mathbb{R}^n$ and $u(\mathbf{x})$ be a continuous function defined in $\Omega \subset \mathbb{R}^n$. Also we note that multi-index notation is used throughout this section.

Let $\mathbf{P}_m(\mathbf{x})$ be the vector such that its elements are made by all polynomials of order less than or equal to m. For example, if $n = 2$ and $m = 2$, then $\mathbf{P}_m(\mathbf{x})$ is $(1, x, y, x^2, xy, y^2)^T$. Suppose we have given node set $\Lambda = \{\mathbf{x}_i \in \Omega \,|\, i = 1, 2, \cdots, NP\}$ on the domain $\Omega \subset \mathbb{R}^n$. With a compactly supported smooth non-negative window function Φ, the resulting shape functions of MLSRK are defined as the following:

$$\phi_i(\mathbf{x}) \equiv \mathbf{e}\, M^{-1}(\mathbf{x})\, \mathbf{P}_m \left(\frac{\mathbf{x}_i - \mathbf{x}}{\rho} \right) \Phi_\rho(\mathbf{x}_i - \mathbf{x}). \tag{1}$$

Here, $\Phi_\rho(\mathbf{x} - \bar{\mathbf{x}}) = \dfrac{1}{\rho^n} \Phi \left(\dfrac{\mathbf{x} - \bar{\mathbf{x}}}{\rho} \right)$. The parameter ρ is called the dilation parameter which determines the support of the shape function, and we will assume that ρ is constant. The matrix M is the moment matrix as the following;

$$M(\mathbf{x}) \equiv \sum_{j=1}^{NP} \mathbf{P}_m \left(\frac{\mathbf{x}_j - \mathbf{x}}{\rho} \right) \mathbf{P}_m^T \left(\frac{\mathbf{x}_j - \mathbf{x}}{\rho} \right) \Phi_\rho(\mathbf{x}_j - \mathbf{x}).$$

Also $\mathbf{e} = (1, 0, \cdots, 0)$ is the unit vector in $\mathbb{R}^{\frac{(n+m)!}{n!m!}}$. For example, if $n = 2$ and $m = 2$, explicit shape functions are the following.

$$\phi_i(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} M^{-1}(\mathbf{x}) \begin{pmatrix} 1 \\ \frac{x_i - x}{\rho} \\ \frac{y_i - y}{\rho} \\ \frac{(x_i - x)^2}{\rho^2} \\ \frac{(x_i - x)(y_i - y)}{\rho^2} \\ \frac{(y_i - y)^2}{\rho^2} \end{pmatrix} \Phi_\rho(\mathbf{x}_i - \mathbf{x}).$$

Since there is a close relation between node distribution and shape functions, we assume that the node set satisfies a certain uniform distribution as the following.

Definition 1. Let $\Lambda = \{\mathbf{x}_i | i = 1, \cdots, NP\}$ be the set of nodes. We say Λ is a *regular node set* if the followings hold.
i) There exists $C_1 > 0$ independent of NP such that

$$\min_i h_{\mathbf{x}_i} \geq C_1 \max_i h_{\mathbf{x}_i} \ ,$$

where $h_{\mathbf{x}_i} = \min_{j \neq i} |\mathbf{x}_i - \mathbf{x}_j|$. We take $h = \min_i h_{\mathbf{x}_i}$ as a characteristic distance of Λ.
ii) Let $\rho = \gamma h$, for some fixed $\gamma > 1$. There exists $C_\gamma > 0$ depending only on γ such that

$$\min_i N(i, \gamma) \geq C_\gamma \max_i N(i, \gamma)$$

where $N(i, \gamma)$ is the number of nodes contained in $B_\rho(\mathbf{x}_i)$. We call γ the dilation ratio and ρ the support radius.

Also we assume that the set of shape function satisfies the following condition.

Definition 2. Let $A = \{\phi_i | i = 1, \cdots, NP\}$ be the set of MLSRK shape functions generated by the window function Φ for the regular node set $\Lambda = \{\mathbf{x}_i | i = 1, \cdots, NP\}$. Then A and Λ are *admissible* if there is a positive constant β_0 such that

$$\sum_{\alpha=1}^{n} \sum_{i,j=1}^{NP} \int_\Omega \phi_i \phi_j \, dx \, a_i^\alpha \, a_j^\alpha \geq \beta_0 \|a\|^2 \tag{2}$$

for all $a^\alpha \in \mathbb{R}^{NP}, \alpha = 1, ..., n$.

The above *regular* condition for node set implies an overlapping condition of shape functions, that is, sufficient number of node points belong to the support of each shape function ϕ_i to ensure the stability of the moment matrix. In short, the *regular* condition implies a certain uniform condition for the node distance and support radius of shape functions. We refer [3] for more details.

For the convergence analysis, we need an interpolation error estimate between the solution space and the projection generated by the set of shape functions. We state the interpolation error estimate theorem and related definitions in [3].

Definition 3. Let $A = \{\phi_i | i = 1, \cdots, NP\}$ be the set of MLSRK shape functions generated by the window function Φ for the regular node set $\Lambda = \{\mathbf{x}_i | i = 1, \cdots, NP\}$. Let $u(\mathbf{x}) \in C^0(\Omega)$ be a function and $\rho > 0$ a real number. We define the discrete projection as

$$\mathcal{R}^m_{\rho,h} \, u(\mathbf{x}) \equiv \sum_{i=1}^{NP} u(\mathbf{x}_i) \, \phi_i(\mathbf{x}) = \sum_{\mathbf{x}_i \in \Lambda(\mathbf{x})} u(\mathbf{x}_i) \, \phi_i(\mathbf{x}) \, ,$$

where $\phi_i(\mathbf{x})$ is the same as (1) and $\Lambda(\mathbf{x}) = \{\mathbf{x}_i \in \Lambda | \mathbf{x} \in \operatorname{supp} \phi_i \cap \bar{\Omega}\}$. Here, m denotes the order of generating polynomial basis $\mathbf{P}_m$.

Remark 1. The notation $\mathcal{R}^m_{\rho,h}$ was used in [14] by Liu et. al. We follow their notation for convenience. The above interpolation exactly reproduces elements in $\mathbf{P}_m(\mathbf{x})$. Such property is called by the m-th order consistency. Henceforth, the term 'shape functions with the m-th order consistency' implies that meshfree shape functions are generated from $\mathbf{P}_m(\mathbf{x})$.

Theorem 1. *(Theorem 2.1 in [3]) Assume the window function $\Phi(\mathbf{x}) \in C_0^m(\mathbb{R}^n)$ and $v(\mathbf{x}) \in C^{m+1}(\overline{\Omega})$, where Ω is a bounded open set in $\mathbb{R}^n$. Let $\Lambda = \{\mathbf{x}_i | i = 1, \cdots, NP\}$ be a regular node set and $A = \{\phi_i | i = 1, \cdots, NP\}$ be the set of admissible shape functions. Suppose the boundary of Ω is smooth and $supp\phi_i \cap \bar{\Omega}$ is convex for each i. If m and p satisfy*

$$m > \frac{n}{p} - 1 \quad ,$$

then the following interpolation estimate holds

$$\|v - \mathcal{R}_{\rho,h}^m v\|_{W^{k,p}(\Omega)} \leq C_k \, \rho^{m+1-k} \, \|v\|_{W^{m+1,p}(\Omega)}, \quad \textit{for all} \quad 0 \leq k \leq m \ . \tag{3}$$

For the analysis of convection terms for the Navier-Stokes equations, we introduce the discrete Sobolev embedding theorem proposed by Choe et. al.[3].

Theorem 2. *(Theorem 2.2 in [3]) We assume $supp\, \phi_i \cap \Omega$ is convex. Suppose $v \in C_0^1(\Omega)$ and the window function $\Phi \in C_0^m(\mathbb{R}^n), m \geq 1$. If $p > n$ holds, then we have the following inequalities*

$$\|\nabla \mathcal{R}_{\rho,h}^m v\|_{L^p} \leq C \|\nabla v\|_{L^p}, \tag{4}$$

$$\sup_{\Omega} |\mathcal{R}_{\rho,h}^m v| \leq C |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|\nabla v\|_{L^p} \ . \tag{5}$$

3. **Applications of the MLSRK Method to the Incompressible Flow.** In this section, we consider the meshfree solutions for the non-stationary incompressible viscous flows by adopting the idea of the mixed formulation. It is well known that meshfree shape functions do not satisfy the Kronecker-δ property. Hence the boundary integral terms are inevitable when using the variational formulation. However, we can choose the basis of the discrete solution space such that they satisfy a certain rate of decay near the boundary of the domain.

In [3], the authors assumed two conditions on the test function space of the velocity field. The window function was chosen with a specific decay rate, and it was named *the proper window function*. The test function space for the velocity consisted of the boundary transformed interior shape functions. The boundary transformed shape functions are linear combinations of shape functions, so that they satisfy the Kronecker-δ property on every boundary node points. As a result, elements of the test function space for velocity satisfy sufficient decay rate near the boundary, $\|\widehat{\phi}_i\|_{L_\infty(\partial\Omega)} \leq c\rho^{2m}$. Then boundary integral terms of variational formulation are negligible in the analysis as we showed in [3]. We refer [3] for details on constructing the test function space for convenience. In this paper, we take the same kind of test function space for velocity as in [3] so that any boundary integrals are negligible through analysis. Henceforth, for simplicity, we omit terms involving boundary integral through the paper. We show a numerical example for the existence of such a basis in the next section.

Assume Ω is a bounded smooth domain in $\mathbb{R}^n$ $(n = 2, 3)$. Let $A^V = \{\phi_i \,|\, i = 1, \cdots, NP\}$ be the set of MLSRK shape functions, generated from a *proper window function* Φ to meet m-th order consistency for a given velocity nodes set $\Lambda^V = \{\mathbf{x}_i^V \,|\, i = 1, \cdots, NP\}$. We notate the set of interior node points $\widehat{\Lambda^V} = \{\mathbf{x}_i^V \,|\, i = 1, \cdots, \widehat{NP}\}$. Also, the boundary transformed interior shape functions are notated as $\widehat{A^V} = \{\widehat{\phi}_i \,|\, i = 1, \cdots, \widehat{NP}\}$. For the pressure, let $A^P = \{\psi_j \,|\, j = 1, \cdots, MP\}$ be the set of MLSRK shape functions generated from a window function Ψ with m-th order consistency for a given pressure nodes set $\Lambda^P = \{\mathbf{x}_j^P \,|\, j = 1, \cdots, MP\}$.

Throughout this section, we assume $\widehat{A^V}$ and A^P satisfy the admissibility condition which is defined in the previous section. In general, window functions Φ and Ψ are different. Here, the $\widehat{\phi}_i$'s and ψ_j's are naturally associated with $\mathbf{x}_i^V \in \widehat{\Lambda^V}$ and $\mathbf{x}_j^P \in \Lambda^P$, respectively. We also assume that the characteristic distance for the pressure node is compatible to the characteristic distance for the velocity node, and so is the characteristic support radius of the shape function. Hence the discrete solution spaces for velocity and pressure with finite time interval $[0,T]$ will be as follows:

$$C^1([0,T]:\left[V_0^h(\Omega)\right]^n) = \left\{ \sum_{i=1}^{\widehat{NP}} \mathbf{u}_i(t)\widehat{\phi}_i(\mathbf{x}) \mid \widehat{\phi}_i \in \widehat{A^V}, \mathbf{u}_i(t) \in C^1[0,T] \right\} \tag{6}$$

$$C^0([0,T]:M^h(\Omega)) = \left\{ \sum_{j=1}^{MP} p_j(t)\psi_j(\mathbf{x}) \mid \psi_j \in A^P, p_j \in C^0[0,T], \int_\Omega P(\mathbf{x})\,d\mathbf{x} = 0 \right\} \tag{7}$$

Note that the transformed velocity shape function satisfies the decay rate $\|\widehat{\phi}_i\|_{L_\infty(\partial\Omega)} \leq c\rho^{2m}$.

Since we are considering different sets of shape functions like the set of velocity shape functions and the set of pressure shape functions, we need to consider two kinds of discrete projections, $\widehat{\mathcal{R}^m_{\rho_V,h_V}}$ for the velocity and $\mathcal{S}^m_{\rho_P,h_P}$ for the pressure as follows

$$\widehat{\mathcal{R}^m_{\rho_V,h_V}}\mathbf{u}(\mathbf{x},t) = \sum_{i=1}^{NP} \mathbf{u}(\mathbf{x}_i^V,t)\,\widehat{\phi}_i(\mathbf{x}),$$

$$\mathcal{S}^m_{\rho_P,h_P}p(\mathbf{x},t) = \sum_{j=1}^{MP} p(\mathbf{x}_j^P,t)\,\psi_j(\mathbf{x}),$$

for any function $\mathbf{u}(\mathbf{x},t) \in C([0,T]:C_0(\Omega))$ and $p(\mathbf{x},t) \in C([0,T]:C(\Omega))$. We are considering compatible node sets Λ_V and Λ_P, i.e, $c\rho_P \leq \rho_V \leq C\rho_P$ and $ch_P \leq h_V \leq Ch_P$. Therefore, the interpolation error for $\widehat{\mathcal{R}^m_{\rho_V,h_V}}$ and $\mathcal{S}^m_{\rho_P,h_P}$ are compatible. Hence for simplicity, we will notate $\widehat{\mathcal{R}^m_{\rho_V,h_V}}$ as $\widehat{\mathcal{R}^m_{\rho,h}}$ and $\mathcal{S}^m_{\rho_P,h_P}$ as $\mathcal{S}^m_{\rho,h}$. We note that the projection error estimate theorem and the discrete Sobolev embedding theorem are valid for $\widehat{\mathcal{R}^m_{\rho,h}}$.(Corollary 2.1, 2.2 in [3]),

We state the MLSRK version of the *inf-sup condition* which was introduced in [3] for further study, while the original *inf-sup condition* was introduced by Babuška [2].

Definition 4. We say a pair of shape function sets $(\widehat{A^V}, A^P)$ satisfy the *inf-sup condition if there exists* $\lambda > 0$ *independent of* ρ *such that*

$$\sup_{\mathbf{U}\in\left[V_0^h(\Omega)\right]^n} \frac{<\operatorname{div}\mathbf{U}, P>}{\|\nabla\mathbf{U}\|_{L^2}} \geq \lambda\|P\|_{L^2}, \tag{8}$$

for all $P \in M^h(\Omega)$.

We note that a sufficient condition, for the pair $(\widehat{A^V}, A^P)$ to meet the inf-sup condition, was shown in [3].

3.1. **Stokes Problem.** In this subsection, we consider the non-stationary incompressible Stokes equations with zero boundary condition. The governing equations are

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}\,, \\ \nabla\cdot\mathbf{u} &= 0\,, \\ \mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega\,, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{9}$$

where the solution $(\mathbf{u}, p) \in L^2(0,\infty : H_0^2(\Omega)) \cap L^\infty(0,\infty : H_0^1(\Omega)) \times L^2(0,\infty : H^1(\Omega)/\mathbb{R})$, the external force $\mathbf{f} \in L^2(0,\infty : L^2(\Omega))$, the initial data $\mathbf{u}_0 \in C_0^{m+1}(\Omega)$ is divergence free, and ν is the kinematic viscosity. Using the MLSRK method, we study the existence of the numerical solution and its convergence to the exact solution.

First we define a pair $(\mathbf{U}, P) = \left(\sum_{i=1}^{\widehat{NP}} \mathbf{u}_i(t)\widehat{\phi}_i(\mathbf{x}), \sum_{i=1}^{MP} p_i(t)\psi_i(\mathbf{x})\right)$ in $C^1([0,T] : \left[V_0^h(\Omega)\right]^n) \times C^0([0,T] : M^h(\Omega))$ is the discrete solution to the Stokes equations (9), if $(\mathbf{U}, P)$ satisfy

$$\begin{aligned} \frac{d}{dt}\int_\Omega \mathbf{U}\mathbf{V} d\mathbf{x} + \nu\int_\Omega \nabla\mathbf{U}\,\nabla\mathbf{V}\,d\mathbf{x} + \int_\Omega \mathbf{V}\,\nabla P\,d\mathbf{x} &= \int_\Omega \mathbf{f}\,\mathbf{V}\,d\mathbf{x}, \\ \int_\Omega \mathbf{U}\cdot\nabla Q\,d\mathbf{x} &= 0, \end{aligned} \tag{10}$$

for all $(\mathbf{V}, Q) \in \mathcal{V}\times\mathcal{W}$ for the given initial data $\mathbf{U}(\mathbf{x},0) = \widehat{\mathcal{R}_{\rho,h}^m}(\mathbf{u}_0)(\mathbf{x})$, where

$$\mathcal{V} = \left[V_0^h(\Omega)\right]^n = \left\{ \sum_{i=1}^{\widehat{NP}} \mathbf{u}_i\widehat{\phi}_i(\mathbf{x}) \mid \widehat{\phi}_i \in \widehat{A^V}, \mathbf{u}_i \in \mathbb{R}^n \right\} \tag{11}$$

$$\mathcal{W} = M^h(\Omega) = \left\{ \sum_{j=1}^{MP} p_j\psi_j(\mathbf{x}) \mid \psi_j \in A^P, p_j \in \mathbb{R}, \int_\Omega P(\mathbf{x})\,d\mathbf{x} = 0 \right\}. \tag{12}$$

Then we have the following theorem for the existence and uniqueness of the discrete Stokes problem. For the stability analysis, the regularity condition for the velocity window function Φ and the pressure window function Ψ is necessary. We suppose that $\Phi \in C_0^{m+1}(\mathbb{R}^n)$ and $\Psi \in C_0^m(\mathbb{R}^n)$ for $m \geq 0$. The velocity window function Φ is differentiable one more time than the pressure window function Ψ. A similar assumption for the velocity element function and pressure element function has been made for the mixed finite element theory of the incompressible Navier-Stokes equations. We note that $\widehat{A^V}$ consist of interior transformed shape functions out of A^V ; the MLSRK shape functions generated from Φ, where Φ is a *proper window function*. We also assume that node sets Λ^V and Λ^P are *regular*, and the sets of shape function, $\widehat{A^V}$ and A^P are *admissible* throughout the paper.

Theorem 3. *We let $\widehat{A^V}$ and A^P be the sets of velocity shape functions and pressure shape functions with m-th order consistency, respectively. Suppose that $(\widehat{A^V}, A^P)$ satisfies the inf-sup condition* (8)*. Then for $\mathbf{f} \in L^2(0,\infty : L^2(\Omega))$, there is a unique discrete solution pair $(\mathbf{U}, P) \in C^1([0,\infty) : \mathcal{V}) \times C([0,\infty) : \mathcal{W})$ to discrete n-dimensional Stokes equations (10).*

Proof. We need to find the velocity coefficient vectors $\bar{\mathbf{u}}^\alpha = [\bar{u}_1^\alpha, \cdots, \bar{u}_{\widehat{NP}}^\alpha]$ for $\alpha = 1, \cdots, n$ and the pressure coefficient vector $\bar{p} = [p_1, \cdots, p_{MP}]^T$.

We define

$$\mathcal{V}_\sigma = \{\phi \in \mathcal{V} | \int_\Omega (\text{div}\phi)\, \psi \, d\mathbf{x} = 0 \quad \text{for all} \quad \psi \in \mathcal{W}\} \tag{13}$$

and

$$\mathcal{W}_\delta = \{\psi \in \mathcal{W} | \int_\Omega (\text{div}\phi)\, \psi \, d\mathbf{x} = 0 \quad \text{for all} \quad \phi \in \mathcal{V}\}. \tag{14}$$

Note that the inf-sup condition of $(\widehat{A^V}, A^P)$ implies $\mathcal{W}_\delta = \{0\}$.

Now we formulate an equivalent problem: Find a solution $\mathbf{U} \in C^1([0, \infty) : \mathcal{V}_\sigma)$ satisfying

$$\frac{d}{dt} \int_\Omega \mathbf{U} \cdot \phi \, d\mathbf{x} + \nu \int_\Omega \nabla \mathbf{U} \nabla \phi \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \phi \, d\mathbf{x} \tag{15}$$

for all $\phi \in \mathcal{V}_\sigma$ and $\mathbf{U}(\mathbf{x}, 0) = \left(\widehat{R^m_{\rho,h}} u_0\right)(\mathbf{x})$. Since we are looking for the velocity coefficient vectors $\bar{\mathbf{u}}^\alpha = [\bar{u}_1^\alpha, \cdots, \bar{u}_{\widehat{NP}}^\alpha]$ for $\alpha = 1, \cdots, n$, we may regard (15) as a system of linear ordinary differential equations with initial data $(\mathbf{U}, P) = (\widehat{\mathcal{R}^m_{\rho,h}\, \mathbf{u}_0}, \mathcal{S}^m_{\rho,h}\, p_0)$. Note that it is known that there is a Stokes solution $(\mathbf{u}, p) \in L^2(0, \infty : H_0^2(\Omega)) \cap L^\infty(0, \infty : H_0^1(\Omega)) \times L^2(0, \infty : H^1(\Omega)/\mathbb{R})$ for $\mathbf{f} \in L^2(0, \infty : L^2(\Omega))$. Then from our admissibility condition (2) of the node distribution and the existence theorem of the ordinary differential equation, there is a unique solution in $C^1([0, \infty) : \mathcal{V}_\sigma)$.

Now we let $\mathcal{P}_\mathbf{U} : \mathcal{V} \to \mathbb{R}$ be

$$\mathcal{P}_\mathbf{U}(\phi) = -\frac{d}{dt} \int_\Omega \mathbf{U} \cdot \phi \, d\mathbf{x} - \nu \int_\Omega \nabla \mathbf{U} \nabla \phi \, d\mathbf{x} + \int_\Omega \mathbf{f} \cdot \phi \, d\mathbf{x}, \tag{16}$$

then $ker(\mathcal{P}_\mathbf{U}) \supseteq \mathcal{V}_\sigma$.

Now define $E : \mathcal{W} \to \mathcal{V}'$ by $< E(\psi), \phi >= -\int_\Omega \phi \nabla \psi \, d\mathbf{x}$ for all $\psi \in \mathcal{W}$, $\phi \in \mathcal{V}$. Then the adjoint of E, $D : \mathcal{V} \to \mathcal{W}'$ is induced by $< \psi, D(\phi) >= \int_\Omega (\text{div}\phi)\, \psi \, d\mathbf{x}$, and the range of E, $R(E) \cong (Ker(D))^\perp \equiv \mathcal{V}/\mathcal{V}_\sigma$. Since $\mathcal{V}_\sigma \subseteq ker(\mathcal{P}_\mathbf{U})$, there exists $P \in \mathcal{W}$ such that

$$\mathcal{P}_\mathbf{U}(\phi) = -\int_\Omega \phi \nabla P \, d\mathbf{x} \,,$$

for all $\phi \in \mathcal{V}$. Since we are assuming $\mathbf{f}$ is continuous in $L^2(\Omega)$ as a function of time, $(\mathbf{U}, P)$ is also $C^1 \times C^0$ in time. Moreover, from the uniqueness of the system of the ordinary differential equations, $(\mathbf{U}, P)$ is unique. □

For the convergence analysis, we need to clarify the H^1 projection from $\mathcal{V}$ to $\mathcal{V}_\sigma$. We let $\mathcal{P}_\sigma : \mathcal{V} \to \mathcal{V}_\sigma$ be the H^1 projection, that is, for given $\phi \in \mathcal{V}$ $\mathcal{P}_\sigma(\phi) \in \mathcal{V}_\sigma$ minimizes

$$\int_\Omega |\nabla(\phi - \mathcal{P}_\sigma(\phi))|^2 \, d\mathbf{x}$$

and we define $\eta(\phi) = \phi - \mathcal{P}_\sigma(\phi)$, for each $\phi \in \mathcal{V}$.

Definition 5. We say $(\widehat{A^V}, A^P)$ is non-degenerate, if for all $\phi \in \mathcal{V}$

$$\sup_{\psi \in \mathcal{W}, ||\psi||_{L^2}=1} \int_\Omega [\text{div}(\eta(\phi))] \, \psi \, d\mathbf{x} \geq \lambda_0 \, ||\eta(\phi)||_{H^1} \tag{17}$$

for some $\lambda_0 > 0$.

We know that $\mathcal{V}$ and $\mathcal{V}_\sigma$ are finite dimensional spaces and $\mathcal{V}_\sigma \subset \mathcal{V}$. For each $\phi \in \mathcal{V}$, we also define a map $\mathcal{D}(\phi) : \mathcal{W} \to \mathbb{R}$ by

$$(\mathcal{D}(\phi))(\psi) = \int_\Omega \psi \, \mathrm{div}\phi \, d\mathbf{x}. \tag{18}$$

So $\mathcal{D}$ maps $\mathcal{V}$ into $\mathcal{W}^{'}$ and $ker(\mathcal{D}) = \mathcal{V}_\sigma$, where $\mathcal{W}^{'}$ is the dual space of $\mathcal{W}$. Then, as we have seen in the proof of the previous theorem,

$$\mathcal{V}/\mathcal{V}_\sigma \cong \mathcal{W}$$

that is $\mathcal{V}/\mathcal{V}_\sigma$ is isometrically isomorphic to $\mathcal{W}$. $\mathcal{V}_\sigma$ is the kernel of $\mathcal{D} : \mathcal{V} \to \mathcal{W}^{'}$ and hence $\mathcal{D}$ induces a natural map from $\mathcal{V}/\mathcal{V}_\sigma$ to $\mathcal{W}^{'}$.

Any vector $\phi \in \mathcal{V}$ can be decomposed uniquely as

$$\phi = \mathcal{P}_\sigma \phi + \eta(\phi). \tag{19}$$

Hence from the inner product structure of H^1, we have

$$\int_\Omega \nabla \mathcal{P}_\sigma \phi \, \nabla \eta(\phi) \, d\mathbf{x} = 0. \tag{20}$$

From the motivation of this orthogonality, we define a new norm $||\cdot||_{div}$ on $\eta(\mathcal{V})$ by

$$||\eta(\phi)||_{div} = \sup_{\psi \in \mathcal{W}, ||\psi||_{L^2}=1} \int_\Omega \psi \, \mathrm{div}\phi \, d\mathbf{x}. \tag{21}$$

The inf-sup condition and the orthogonality imply $||\cdot||_{div}$ is well defined norm on $\eta(\mathcal{V})$. Now we define a norm $||\cdot||_{\mathcal{V}}$ by

$$||\phi||_{\mathcal{V}} = ||\mathcal{P}_\sigma \phi||_{H^1} + ||\eta(\phi)||_{div}. \tag{22}$$

We claim that $||\cdot||_{H^1}$ is equivalent to $||\cdot||_{\mathcal{V}}$. From the orthogonality of $\mathcal{P}_\sigma(\phi)$ and $\eta(\phi)$ in H^1, we get

$$||\mathcal{P}_\sigma(\phi)||_{H^1} \le ||\phi||_{H^1}. \tag{23}$$

Considering the definition of $||\phi||_{div}$ and Hölder inequality, we obtain

$$||\eta(\phi)||_{div} = \sup_{\psi \in \mathcal{W}, ||\psi||_{L^2}=1} \int_\Omega \psi \, \mathrm{div}\phi \, d\mathbf{x} \le ||\mathrm{div}\phi||_{L^2} \le ||\phi||_{H^1} \tag{24}$$

and

$$||\mathcal{P}_\sigma(\phi)||_{H^1} + ||\eta(\phi)||_{div} \le 2||\phi||_{H^1}. \tag{25}$$

For the opposite direction, we need to show

$$||\nabla \eta(\phi)||_{L^2} \le c||\eta(\phi)||_{div}. \tag{26}$$

And this follows from the non-degeneracy of $(\widehat{A^V}, A^P)$. Indeed, we have

$$||\nabla \eta(\phi)||_{L^2} \le \frac{1}{\lambda_0} \sup_{\psi \in \mathcal{W}, ||\psi||_{L^2}=1} \int_\Omega \psi \mathrm{div}\phi \, d\mathbf{x} = \frac{1}{\lambda_0} ||\eta(\phi)||_{div}. \tag{27}$$

Lemma 1. *Suppose $(\widehat{A^V}, A^P)$ is non-degenerate and satisfies the inf-sup condition, then for all $\phi \in \mathcal{V}$, we have*

$$\lambda_0 ||\phi||_{H^1} \le ||\mathcal{P}_\sigma(\phi)||_{H^1} + ||\eta(\phi)||_{div} \le 2||\phi||_{H^1}. \tag{28}$$

As a corollary, we obtain the projection error estimate for $\mathcal{P}_\sigma$ on $\mathcal{V}$. Indeed, for a given divergence free vector $\mathbf{u} \in C_0^{m+1}(\Omega)$, we take $\phi = \widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}$ in (28). Then, $\mathcal{P}_\sigma(\phi) = 0$, and we obtain

$$\begin{aligned} \|\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}\|_{H^1} &\le \frac{C}{\lambda_0}\|\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}\|_{div} \\ &= \frac{C}{\lambda_0}\sup_{\psi\in\mathcal{W},\|\psi\|_{L^2}=1}\int_\Omega \mathrm{div}(\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u} - \mathbf{u})\,\psi\,d\mathbf{x} \\ &\le \frac{C}{\lambda_0}\|\mathbf{u} - \widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}\|_{H^1} \le C(\lambda_0)\rho^m\|\mathbf{u}\|_{H^{m+1}} \quad . \end{aligned} \tag{29}$$

Corollary 1. *Suppose $\mathbf{u} \in C_0^{m+1}(\Omega)$ and $div(\mathbf{u}) = 0$, then we have*

$$\|\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}\|_{H^1} \le C(\lambda_0)\rho^m\|\mathbf{u}\|_{H^{m+1}}. \tag{30}$$

Now we are ready to show the error estimate of the MLSRK scheme for the Stokes equations.

Theorem 4. *Let $\widehat{A^V}$ and A^P be the sets of the velocity shape functions and the pressure shape functions, respectively. Suppose $(\widehat{A^V}, A^P)$ is non-degenerate and satisfies inf-sup condition. Let $(\mathbf{u}, p) \in L^2(0,\infty : H_0^2(\Omega)) \cap L^\infty(0,\infty : H_0^1(\Omega)) \times L^2(0,\infty : H^1(\Omega)/\mathbb{R})$ be the solution of the Stokes equations* (9) *for $\mathbf{f} \in L^2(0,\infty : L^2(\Omega))$ and $(\mathbf{U}, P) \in C^1([0,T] : \mathcal{V}) \times C^0([0,T] : \mathcal{W})$ be the MLSRK solution of the discrete Stokes equation* (10)*. Then the following error estimates hold.*

$$\begin{aligned} &\|\mathbf{U} - \mathbf{u}\|^2_{L^2}(T) + \int_0^T \|\nabla(\mathbf{U} - \mathbf{u})\|^2_{L^2}(t)\,dt \\ \le & C\rho^{2m}\left(\|\mathbf{u}_0\|^2_{H^{m+1}} + \|\mathbf{u}\|^2_{H^{m+1}}(T) + \int_0^T \|\mathbf{u}_t\|^2_{H^{m+1}}(t) + \|\mathbf{u}\|^2_{H^{m+1}}(t) + \|p\|^2_{H^m}\,dt\right). \end{aligned} \tag{31}$$

Proof. We have the error equations

$$\begin{aligned} \int_\Omega (\mathbf{U} - \mathbf{u})_t\,\phi\,d\mathbf{x} + \nu\int_\Omega \nabla(\mathbf{U} - \mathbf{u})\cdot\nabla\phi\,d\mathbf{x} + \int_\Omega (P - p)\,\mathrm{div}\phi\,d\mathbf{x} = 0\,, \\ \int_\Omega \psi\,\mathrm{div}(\mathbf{U} - \mathbf{u})\,d\mathbf{x} = 0, \end{aligned} \tag{32}$$

for all $\phi \in \mathcal{V}$ and $\psi \in \mathcal{W}$. Now we take $\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}$ as a test function to our error equations, and we obtain

$$\begin{aligned} \int_\Omega (\mathbf{U} - \mathbf{u})_t\,(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x} + \nu\int_\Omega \nabla(\mathbf{U} - \mathbf{u})\cdot\nabla(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x} \\ + \int_\Omega (P - p)\,\mathrm{div}(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x} = 0. \end{aligned} \tag{33}$$

We add and subtract $\mathbf{u}$ in $\mathbf{U}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}$. Then, integrating in time, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\mathbf{U}-\mathbf{u}|^2(\mathbf{x},T)\,d\mathbf{x}+\nu\int_0^T\int_\Omega |\nabla(\mathbf{U}-\mathbf{u})|^2\,d\mathbf{x}\,dt\\
=&\frac{1}{2}\int_\Omega |\mathbf{U}-\mathbf{u}|^2(\mathbf{x},0)\,d\mathbf{x}-\int_0^T\int_\Omega(\mathbf{U}-\mathbf{u})_t(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x}\,dt\\
&-\nu\int_0^T\int_\Omega \nabla(\mathbf{U}-\mathbf{u})\nabla(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x}\,dt\\
&-\int_0^T\int_\Omega (P-p)\mathrm{div}(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x}\,dt\\
=&\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.
\end{aligned}$$

Since $\mathbf{U}(\mathbf{x},0)=\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}_0(\mathbf{x})$, we have

$$|\mathrm{I}|\le C\rho^{2m}||u_0||^2_{H^{m+1}}.$$

From the integration by parts on time, we get

$$\begin{aligned}
\mathrm{II}=&-\int_\Omega(\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(\mathbf{x},T)\,d\mathbf{x}+\int_\Omega(\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(\mathbf{x},0)\,d\mathbf{x}\\
&+\int_0^T\int_\Omega(\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})_t\,d\mathbf{x}\,dt.
\end{aligned}$$

From Hölder inequality, the interpolation estimate (3) and the projection error estimate to the divergence free space (30), we have

$$\begin{aligned}
&\int_\Omega(\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(\mathbf{x},T)\,d\mathbf{x}\le\frac{1}{8}\int_\Omega|\mathbf{U}-\mathbf{u}|^2(x,T)\,d\mathbf{x}\\
&+C\int_\Omega|\mathbf{u}-\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}|^2(x,T)\,d\mathbf{x}+C\int_\Omega|\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}|^2(\mathbf{x},T)\,d\mathbf{x}\\
\le&\frac{1}{8}\int_\Omega\left|\mathbf{U}-\mathbf{u}\right|^2(\mathbf{x},T)\,d\mathbf{x}+C\rho^{2m}\|u\|^2_{H^{m+1}}(T).
\end{aligned}$$

We know that the projection $\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}$ commutes with the time differentiation. Therefore we get

$$\begin{aligned}
&\left|\int_0^T\int_\Omega(\mathbf{U}-\mathbf{u})\,(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})_t\,d\mathbf{x}\,dt\right|\\
\le&\ \varepsilon_0\int_0^T\int_\Omega\left|\mathbf{U}-\mathbf{u}\right|^2\,d\mathbf{x}\,dt+C\int_0^T\int_\Omega\left|\mathbf{u}_t-\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}_t\right|^2\,d\mathbf{x}\,dt\\
&+C\int_0^T\int_\Omega\left|\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}_t-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}_t\right|^2\,d\mathbf{x}\,dt\\
\le&\ \varepsilon_0\left|\Omega\right|^{\frac{2}{n}}\int_0^T\int_\Omega\left|\nabla\mathbf{U}-\nabla\mathbf{u}\right|^2\,d\mathbf{x}\,dt+C\int_0^T\int_\Omega\left|\mathbf{u}_t-\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}_t\right|^2\,d\mathbf{x}\,dt\\
&+C\int_0^T\int_\Omega\left|\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}_t-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}_t\right|^2\,d\mathbf{x}\,dt\\
\le&\ \frac{\nu}{8}\int_0^T\int_\Omega\left|\nabla\mathbf{U}-\nabla\mathbf{u}\right|^2\,d\mathbf{x}\,dt+C\rho^{2m}\int_0^T\|u_t\|^2_{H^{m+1}}(t)\,dt
\end{aligned}$$

for sufficiently small ε_0. Here we used the Poincáre inequality. From the Hölder inequality, we have

$$\begin{aligned}|\mathrm{III}| \leq &\frac{\nu}{8}\int_0^T\int_\Omega |\nabla(\mathbf{U}-\mathbf{u})|^2\,d\mathbf{x}\,dt + \int_0^T\int_\Omega |\nabla(\mathbf{u}-\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})|^2\,d\mathbf{x}\,dt \\ &+ \int_0^T\int_\Omega |\nabla(\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})|^2\,d\mathbf{x}\,dt \\ \leq &\frac{\nu}{8}\int_0^T\int_\Omega |\nabla(\mathbf{U}-\mathbf{u})|^2\,d\mathbf{x}\,dt + C\rho^{2m}\int_0^T ||u||^2_{H^{m+1}}(t)\,dt.\end{aligned}$$

Finally we have

$$\begin{aligned}\mathrm{IV} =& \int_0^T\int_\Omega (P-\mathcal{S}^m_{\rho,h}p)\,\mathrm{div}(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x}\,dt \\ &+ \int_0^T\int_\Omega (\mathcal{S}^m_{\rho,h}p-p)\,\mathrm{div}(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x}\,dt.\end{aligned}$$

But from the definition of the projection $\mathcal{P}_\sigma$, we have $\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u} \in \mathcal{V}_\sigma$ and

$$\int_0^T\int_\Omega |\nabla(\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})|^2\,d\mathbf{x}\,dt = 0.$$

From the interpolation theorem, we get

$$\begin{aligned}|\mathrm{IV}| \leq &C\int_0^T\int_\Omega |\mathcal{S}^m_{\rho,h}p-p|^2\,d\mathbf{x}\,dt + C\int_0^T\int_\Omega |\nabla(\mathbf{u}-\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})|^2\,d\mathbf{x}\,dt \\ &+ C\int_0^T\int_\Omega |\nabla(\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})|^2\,d\mathbf{x}\,dt \\ \leq &C\rho^{2m}\left(\int_0^T ||p||^2_{H^m}(t)\,dt + \int_0^T ||\mathbf{u}||^2_{H^{m+1}}(t)\,dt\right).\end{aligned}$$

Therefore, combining all the estimates, we get the proof done for (31). □

3.2. **Navier-Stokes Problem.** In this subsection we show the existence and error estimates for the MLSRK solutions of the following non-stationary incompressible Navier-Stokes equations.

$$\begin{aligned}\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f}\,, \\ \nabla\cdot\mathbf{u} &= 0\,, \\ \mathbf{u}(0,x) &= \mathbf{u}_0(x) \quad \text{in } \Omega\,, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega,\end{aligned} \tag{34}$$

where the initial data $\mathbf{u}_0$ is divergence free, the external force $\mathbf{f}$ is a given function with appropriate regularity in $\bar{\Omega}$ and ν is the viscosity. We assume that $(\mathbf{u},p) \in L^2(0,\infty : H^2_0(\Omega)) \cap L^\infty(0,\infty : H^1_0(\Omega)) \times L^2(0,\infty : H^1(\Omega)/\mathbb{R})$. The function space $H^1(\Omega)/\mathbb{R}$ is the set of all $H^1(\Omega)-$functions with zero mean in Ω. The necessary regularity of $\mathbf{f}$ will be assumed in each appearance. We assume the same hypothesis for the velocity and pressure nodes as the admissibility, non-degeneracy and the inf-sup condition in previous section.

We discretize (34) in the following way:

$$\frac{d}{dt}\int_\Omega \mathbf{U}\cdot\mathbf{V}\,d\mathbf{x}+\nu\int_\Omega \nabla\mathbf{U}\nabla\mathbf{V}\,d\mathbf{x}\;+\;\frac{1}{2}\left(\int_\Omega (\mathbf{U}\cdot\nabla)\,\mathbf{U}\mathbf{V}\,d\mathbf{x}-\int_\Omega (\mathbf{U}\cdot\nabla)\,\mathbf{V}\mathbf{U}\,d\mathbf{x}\right)$$
$$+\int_\Omega \nabla P\mathbf{V}\,d\mathbf{x}\;=\;\int_\Omega \mathbf{f}\mathbf{V}\,d\mathbf{x} \tag{35}$$
$$\int_\Omega \mathbf{U}\cdot\nabla Q\,d\mathbf{x}\;=\;0$$

for all $(\mathbf{V},Q)\in\mathcal{V}\times\mathcal{W}$. In the continuous case,

$$\frac{1}{2}(\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\mathbf{v}\,d\mathbf{x}-\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{v}\mathbf{u}\,d\mathbf{x})=\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\mathbf{v}\,d\mathbf{x} \tag{36}$$

since $\nabla\cdot\mathbf{u}=0$. Using (36), we have a priori estimate (38) for the following existence theorem.

Theorem 5. *We let* $\widehat{A^V}$ *and* A^P *be the sets of the velocity shape functions and the pressure shape functions, respectively. Suppose that* $(\widehat{A^V},A^P)$ *satisfies the inf-sup condition. Then for* $\mathbf{f}\in C([0,\infty):C^1(\Omega))$ *there is a unique discrete solution pair* $(\mathbf{U},P)\in C^1([0,\infty):\mathcal{V})\times C([0,\infty):\mathcal{W})$ *for the discrete* n*-dimensional Navier-Stokes equations* (35)*, where* $\mathcal{V}$ *and* $\mathcal{W}$ *are defined as* (11)*,* (12)*.*

Proof. Following the same way as in the Stokes equation, we employ $\mathcal{V}_\sigma=\{\phi\in\mathcal{V}|\int_\Omega(\mathrm{div}\phi)\psi=0\quad\text{for all}\quad\psi\in\mathcal{W}\}$ and $\mathcal{W}_\delta=\{\psi\in\mathcal{W}|\int_\Omega(\mathrm{div}\phi)\psi=0,\quad\text{for all}\quad\phi\in\mathcal{V}\}$. Now consider an equivalent problem: Find a solution $\mathbf{U}\in C^1([0,\infty):\mathcal{V}_\sigma)$ satisfying

$$\frac{d}{dt}\int_\Omega \mathbf{U}\cdot\phi\,d\mathbf{x}+\nu\int_\Omega\nabla\mathbf{U}\nabla\phi\,d\mathbf{x} \tag{37}$$
$$+\frac{1}{2}\left(\int_\Omega(\mathbf{U}\cdot\nabla)\mathbf{U}\phi\,d\mathbf{x}-\int_\Omega(\mathbf{U}\cdot\nabla)\phi\mathbf{U}\,d\mathbf{x}\right)=\int_\Omega\mathbf{f}\cdot\phi\,d\mathbf{x},$$

for all $\phi\in\mathcal{V}_\sigma$ and $\mathbf{U}(\mathbf{x},0)=\left(\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}_0\right)(\mathbf{x})$. The equation (37) is a system of nonlinear ordinary differential equations. Unlike the Stokes equations, the admissibility assumption guarantee only the local existence of the solution near $t=0$. The global existence of the solution for the above equations is obtained from the following a priori estimate. Taking $\mathbf{U}$ as a test function to (37) and integrating for the time, we obtain

$$\int_\Omega|\mathbf{U}(t,\cdot)|^2\,d\mathbf{x}+\nu\int_0^t\int_\Omega|\nabla\mathbf{U}|^2\,d\mathbf{x}\,dt\le\int_\Omega|\mathbf{U}(0,\cdot)|^2\,d\mathbf{x}+\int_0^t\int_\Omega\mathbf{f}\cdot\mathbf{U}\,d\mathbf{x}dt$$
$$\le\int_\Omega|\mathbf{U}(0,\cdot)|^2\,d\mathbf{x}+C(\varepsilon)\int_0^t\int_\Omega|\mathbf{f}|^2\,d\mathbf{x}\,dt+\varepsilon\int_0^t\int_\Omega|\mathbf{U}|^2\,d\mathbf{x}\,dt \tag{38}$$
$$\le\int_\Omega|\mathbf{U}(0,\cdot)|^2\,d\mathbf{x}+C(\varepsilon)\int_0^t\int_\Omega|\mathbf{f}|^2\,d\mathbf{x}\,dt+\varepsilon C(\Omega)\int_0^t\int_\Omega|\nabla\mathbf{U}|^2\,d\mathbf{x}\,dt\,.$$

In the last inequality, we used the Poincáre inequality. Now choosing small ε, we have

$$\int_\Omega|\mathbf{U}(t,\cdot)|^2\,d\mathbf{x}+\nu\int_0^t\int_\Omega|\nabla\mathbf{U}|^2\,d\mathbf{x}\,dt\le C\left(\int_0^t\int_\Omega|\mathbf{f}|^2\,d\mathbf{x}\,dt+\int_\Omega|\mathbf{U}(0,\cdot)|^2\,d\mathbf{x}\right).$$

We can find the pressure $P \in C([0,\infty) : \mathcal{W})$ satisfying

$$-\int \phi \nabla P \, d\mathbf{x} = \int_\Omega \mathbf{U}_t \cdot \phi + \nu \nabla \mathbf{U} \cdot \nabla \phi + \frac{1}{2}\left\{(\mathbf{U}\cdot\nabla)\mathbf{U}\phi - (\mathbf{U}\cdot\nabla)\phi\mathbf{U}\right\} - \mathbf{f}\phi_k \, d\mathbf{x},$$

by following the same way as in the Stokes equations. Since we are assuming $\mathbf{f}$ is continuous in $L^2(\Omega)$ as a function of time, $(\mathbf{U}, P)$ is also $C^1 \times C^0$ in time. Moreover, from the local uniqueness of the system of ordinary differential equations, $(\mathbf{U}, P)$ is unique. □

For the stability analysis, we employ the following error equations by comparing continuous Navier-Stokes equations (34) and discrete Navier-Stokes equations (35).

$$\begin{aligned} &\int_\Omega (\mathbf{U}-\mathbf{u})_t \phi \, d\mathbf{x} + \nu \int_\Omega \nabla(\mathbf{U}-\mathbf{u}) \cdot \nabla\phi \, d\mathbf{x} \\ &+ \frac{1}{2}\int_\Omega \{(\mathbf{U}-\mathbf{u})\cdot\nabla\mathbf{U}\phi + u\cdot\nabla(\mathbf{U}-\mathbf{u})\phi\} \, d\mathbf{x} \\ &+ \frac{1}{2}\int_\Omega \{(\mathbf{U}-\mathbf{u})\cdot\phi\nabla\mathbf{U} + u\cdot\nabla\phi(\mathbf{U}-\mathbf{u})\} \, d\mathbf{x} - \int_\Omega (\mathbf{U}-\mathbf{u})\mathrm{div}\phi \, d\mathbf{x} = 0, \\ &\int_\Omega \psi \mathrm{div}(\mathbf{U}-\mathbf{u}) \, d\mathbf{x} = 0, \end{aligned} \tag{39}$$

for all $\phi \in \mathcal{V}$, $\psi \in \mathcal{W}$. Here, we need to be careful since the sufficient regularity of $\mathbf{u}$ is not known. But we take $\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u} \in \mathcal{V}$ as a test function to (39), and we obtain

$$\begin{aligned} &\int_\Omega (\mathbf{U}-\mathbf{u})_t(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}) \, d\mathbf{x} + \nu \int_\Omega \nabla(\mathbf{U}-\mathbf{u}) \cdot \nabla(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}) \, d\mathbf{x} \\ &+ \frac{1}{2}\int_\Omega \{(\mathbf{U}-\mathbf{u})\cdot\nabla\mathbf{U}(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}) + u\cdot\nabla(\mathbf{U}-\mathbf{u})(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\} \, d\mathbf{x} \\ &+ \frac{1}{2}\int_\Omega \{(\mathbf{U}-\mathbf{u})\cdot(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\nabla\mathbf{U} + u\cdot\nabla(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(\mathbf{U}-\mathbf{u})\} \, d\mathbf{x} \\ &- \int_\Omega (P-p)\mathrm{div}(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}) \, d\mathbf{x} = 0. \end{aligned} \tag{40}$$

As in the case of the Stokes equations, we will estimate each term of (40). For the first term of (40), we have

$$\begin{aligned} &\int_\Omega (\mathbf{U}-\mathbf{u})_t(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}) \, d\mathbf{x} \\ =&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{U}-\mathbf{u}|^2 \, d\mathbf{x} + \int_\Omega (\mathbf{U}-\mathbf{u})_t(\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}) \, d\mathbf{x} \\ =&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{U}-\mathbf{u}|^2 \, d\mathbf{x} + \frac{d}{dt}\int_\Omega (\mathbf{U}-\mathbf{u})(\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}) \, d\mathbf{x} \\ &- \int_\Omega (\mathbf{U}-\mathbf{u})(\mathbf{u}_t - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}_t) \, d\mathbf{x}. \end{aligned} \tag{41}$$

For the second term of (40), by adding and subtracting $\mathbf{u}$, we have

$$\begin{aligned} &\int_\Omega \nabla(\mathbf{U}-\mathbf{u}) \cdot \nabla(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}) \, d\mathbf{x} \\ =&\int_\Omega |\nabla(\mathbf{U}-\mathbf{u})|^2 \, d\mathbf{x} + \int_\Omega \nabla(\mathbf{U}-\mathbf{u}) \cdot \nabla(\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}) \, d\mathbf{x}. \end{aligned} \tag{42}$$

For the other terms which are not involving pressure, since cancelations occur, we have the followings.

$$\begin{aligned}
&\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla\mathbf{U}(\mathbf{U}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x}+\int_\Omega \mathbf{u}\cdot\nabla(\mathbf{U}-\mathbf{u})(\mathbf{U}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x} \qquad (43)\\
&\quad-\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla(\mathbf{U}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\mathbf{U}\,d\mathbf{x}-\int_\Omega \mathbf{u}\cdot\nabla(\mathbf{U}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(\mathbf{U}-\mathbf{u})\,d\mathbf{x}\\
&=\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla\mathbf{u}(\mathbf{U}-\mathbf{u})\,d\mathbf{x}+\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla\mathbf{U}(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x}\\
&\quad+\int_\Omega \mathbf{u}\cdot\nabla(\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x}-\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(\mathbf{U}-\mathbf{u})\,d\mathbf{x}\\
&\quad-\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla(\mathbf{U}-\mathbf{u})\mathbf{u}\,d\mathbf{x}-\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\mathbf{u}\,d\mathbf{x}\\
&\quad-\int_\Omega \mathbf{u}\cdot\nabla(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(\mathbf{U}-\mathbf{u})\,d\mathbf{x}\\
&=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII}.
\end{aligned}$$

From the Hölder inequality, the Sobolev inequality and the Young's inequality, we have

$$\begin{aligned}
|\mathrm{I}| &= \left|\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla\mathbf{u}(\mathbf{U}-\mathbf{u})\,d\mathbf{x}\right| \qquad (44)\\
&\le C\|\mathbf{U}-\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^3}\|\mathbf{U}-\mathbf{u}\|_{L^6}\\
&\le C\|\mathbf{U}-\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}\|\nabla(\mathbf{U}-\mathbf{u})\|_{L^2}\\
&\le C\varepsilon\|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2}+\frac{C}{\varepsilon}\|\mathbf{u}\|^2_{H^2}\|\mathbf{U}-\mathbf{u}\|^2_{L^2},
\end{aligned}$$

for some $\varepsilon>0$. We have a similar estimate for II, for some $\varepsilon>0$,

$$\begin{aligned}
|\mathrm{II}| &= \left|\int_\Omega (\mathbf{U}-\mathbf{u})\cdot\nabla\mathbf{U}(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})\,d\mathbf{x}\right| \qquad (45)\\
&\le C\|\mathbf{U}-\mathbf{u}\|_{L^6}\|\nabla\mathbf{U}\|_{L^3}\|\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}\|_{L^2}\\
&\le C\|\nabla(\mathbf{U}-\mathbf{u})\|_{L^2}\|\nabla\mathbf{U}\|_{L^3}\{\|\mathbf{u}-\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}\|_{L^2}+\|\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}\|_{L^2}\}\\
&\le C\varepsilon\|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2}+\frac{C\rho^{2m}}{\varepsilon}\|\mathbf{U}\|^2_{H^2}\|\mathbf{u}\|^2_{H^{m+1}}.
\end{aligned}$$

Here, we have used the interpolation estimate (3) and the projection estimate (30). In detail, we used

$$\|\mathbf{u}-\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}\|_{L^3}\le C\rho^{m+1}\|\mathbf{u}\|_{H^{m+1}} \qquad (46)$$

and

$$\begin{aligned}
&\|\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}\|_{L^2}\le C\|\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}\|_{H^1} \qquad (47)\\
&\le\frac{C}{\lambda_0}\|\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}\|_{\mathrm{div}}=\frac{C}{\lambda_0}\sup_{\psi\in W,\|\psi\|_{L^2}=1}\int_\Omega\psi\,\mathrm{div}(\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}-\mathbf{u})\,d\mathbf{x}\\
&\le\frac{C}{\lambda_0}\|\mathbf{u}-\widehat{\mathcal{R}^m_{\rho,h}}\,\mathbf{u}\|_{H^1}\le\frac{C}{\lambda_0}\rho^m\|\mathbf{u}\|_{H^{m+1}}.
\end{aligned}$$

Using similar arguments as above, we have the following estimates,

$$
\begin{aligned}
|\mathrm{III}| &= \left| \int_\Omega \mathbf{u} \cdot \nabla(\mathbf{U}-\mathbf{u})(\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})\, d\mathbf{x} \right| \\
&\le C\varepsilon \|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2} + \frac{C\rho^{2m}}{\varepsilon} \|\mathbf{u}\|^2_{H^{m+1}} \|\mathbf{u}\|^2_{H^1} \,,
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
|\mathrm{IV}| &= \left| \int_\Omega (\mathbf{U}-\mathbf{u}) \cdot \nabla(\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})(\mathbf{U}-\mathbf{u})\, d\mathbf{x} \right| \\
&\le C\rho^m \|\mathbf{u}\|_{H^{m+1}} \|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2} \,.
\end{aligned} \tag{49}
$$

Since L_∞-norm is bounded by the H_2-norm from the Sobolev inequality, we have

$$
\begin{aligned}
|\mathrm{V}| &= \left| \int_\Omega (\mathbf{U}-\mathbf{u}) \cdot \nabla(\mathbf{U}-\mathbf{u})\mathbf{u}\, d\mathbf{x} \right| \\
&\le C\varepsilon \|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2} + \frac{C}{\varepsilon} \|\mathbf{U}-\mathbf{u}\|^2_{L^2} \|\mathbf{u}\|^2_{H^2} \,.
\end{aligned} \tag{50}
$$

From the interpolation estimate (3) and the projection estimate (30), we have

$$
\begin{aligned}
|\mathrm{VI}| &= \left| \int_\Omega (\mathbf{U}-\mathbf{u}) \cdot \nabla(\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})\mathbf{u}\, d\mathbf{x} \right| \\
&\le C\varepsilon \|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2} + \frac{C\rho^{2m}}{\varepsilon} \|\mathbf{u}\|^2_{H^{m+1}} \|\mathbf{u}\|^2_{H^1} \,,
\end{aligned} \tag{51}
$$

and

$$
\begin{aligned}
|\mathrm{VII}| &= \left| \int_\Omega \mathbf{u} \cdot \nabla(\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})(\mathbf{U}-\mathbf{u})\, d\mathbf{x} \right| \\
&\le C\varepsilon \|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2} + \frac{C\rho^{2m}}{\varepsilon} \|\mathbf{u}\|^2_{H^{m+1}} \|\mathbf{u}\|^2_{H^1} \,.
\end{aligned} \tag{52}
$$

For the last term of the left hand side of equation (40), we have

$$
\begin{aligned}
&\int_\Omega (P-p)\mathrm{div}(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})\, d\mathbf{x} \\
=&\int_\Omega (P - \mathcal{S}^m_{\rho,h}\, p)\mathrm{div}(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})\, d\mathbf{x} + \int_\Omega (\mathcal{S}^m_{\rho,h}\, p - p)\mathrm{div}(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})\, d\mathbf{x} \\
=&\int_\Omega (\mathcal{S}^m_{\rho,h}\, p - p)\mathrm{div}(\mathbf{U} - \mathbf{u})\, d\mathbf{x} + \int_\Omega (\mathcal{S}^m_{\rho,h}\, p - p)\mathrm{div}(\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})\, d\mathbf{x}.
\end{aligned} \tag{53}
$$

Note that $P - \mathcal{S}^m_{\rho,h}\, p \in \mathcal{W}$ and $\mathrm{div}(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u}) \in \mathcal{V}_\sigma$. Hence we have

$$
\begin{aligned}
&\left| \int_\Omega (P-p)\mathrm{div}(\mathbf{U} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})\, d\mathbf{x} \right| \\
\le& C\varepsilon \|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2} + (\frac{C}{\varepsilon} + 1)\|\mathcal{S}^m_{\rho,h}\, p - p\|^2_{L^2} + \left\|\nabla(\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u})\right\|^2_{L^2} \\
\le& C\varepsilon \|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2} + (\frac{C}{\varepsilon} + 1)\rho^{2(m+1)} \|p\|^2_{H^{m+1}} + C\rho^{2m} \|\mathbf{u}\|^2_{H^{m+1}}.
\end{aligned} \tag{54}
$$

Therefore combining all previous estimates from (41) to (54) and choosing a sufficiently small $\varepsilon > 0$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\mathbf{U}-\mathbf{u}|^2(t)\,d\mathbf{x} + \nu\,(1 - C_0\rho^m \|\mathbf{u}\|_{H^{m+1}}(t))\int_\Omega |\nabla(\mathbf{U}-\mathbf{u})|^2(t)\,d\mathbf{x} \qquad (55)\\
\le &-\frac{d}{dt}\int_\Omega (\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(t)\,d\mathbf{x} + C_1\int_\Omega \left|\mathbf{u}_t - \mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u}_t\right|^2(t)\,d\mathbf{x}\\
&+ C_2\|\mathbf{U}-\mathbf{u}\|^2_{L^2}(t)\left(1+\|\mathbf{u}\|^2_{H^2}(t)\right)\\
&+ C_3\rho^{2m}\|\mathbf{u}\|^2_{H^{m+1}}(t)\left(1+\|\mathbf{U}\|^2_{H^2}(t)+\|\mathbf{u}\|^2_{H^2}(t)\right).
\end{aligned}
$$

Now let the L^2-norm of the error be the following,

$$
E(t) = \int_\Omega |\mathbf{U}-\mathbf{u}|^2(t)\,d\mathbf{x}\,, \tag{56}
$$

and let the integrating factor be the following

$$
\alpha(t) = C_2\int_0^t \left(1+\|\mathbf{u}\|^2_{H^2}(t)\right)\,dt, \tag{57}
$$

then multiplying the integrating factor $e^{-\alpha(t)}$, (55) becomes

$$
\begin{aligned}
&\frac{d}{dt}\left(e^{-\alpha(t)}E(t)\right) + \nu\,(1 - C_0\rho^m\|\mathbf{u}\|_{H^{m+1}}(t))\,e^{-\alpha(t)}\int_\Omega |\nabla(\mathbf{U}-\mathbf{u})|^2(t)\,d\mathbf{x} \qquad (58)\\
\le &\frac{d}{dt}\left(-e^{-\alpha(t)}\int_\Omega (\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(t)\,d\mathbf{x}\right)\\
&- \alpha^{'}(t)e^{-\alpha(t)}\int_\Omega (\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(t)\,d\mathbf{x}\\
&+ C_4\rho^{2m}e^{-\alpha(t)}\{\|\mathbf{u}_t\|^2_{H^{m+1}}(t) + \|\mathbf{u}\|^2_{H^{m+1}}(t)\left(1+\|\mathbf{U}\|^2_{H^2}(t)+\|\mathbf{u}\|^2_{H^2}(t)\right)\}\,.
\end{aligned}
$$

Hence integrating (58), we get

$$
\begin{aligned}
&e^{-\alpha(T)}E(T) + \int_0^T\int_\Omega \nu\,(1 - C_0\rho^m\|\mathbf{u}\|_{H^{m+1}}(t))\,e^{-\alpha(t)}|\nabla(\mathbf{U}-\mathbf{u})|^2(t)\,d\mathbf{x}\,dt\\
\le &\,E(0) - e^{-\alpha(T)}\int_\Omega (\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(T)\,d\mathbf{x} \qquad (59)\\
&+ e^{-\alpha(0)}\int_\Omega (\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(0)\,d\mathbf{x}\\
&- \int_0^T\int_\Omega \alpha^{'}(t)e^{-\alpha(t)}(\mathbf{U}-\mathbf{u})(\mathbf{u}-\mathcal{P}_\sigma\widehat{\mathcal{R}^m_{\rho,h}}\mathbf{u})(t)\,d\mathbf{x}\,dt\\
&+ C_4\rho^{2m}\int_0^T e^{-\alpha(t)}\{\|\mathbf{u}_t\|^2_{H^{m+1}}(t) + \|\mathbf{u}\|^2_{H^{m+1}}(t)\left(1+\|\mathbf{U}\|^2_{H^1}(t)+\|\mathbf{u}\|^2_{H^2}(t)\right)\}\,dt\,.
\end{aligned}
$$

Applying the Young's inequality and the interpolation estimate (3) and the projection error estimate to the divergence free space (30) to the terms on the righthand side of (59), we have

$$
\begin{aligned}
& e^{-\alpha(T)}E(T) + \int_0^T \int_\Omega \nu \left(1 - C_0\rho^m \|\mathbf{u}\|_{H^{m+1}}(t)\right) e^{-\alpha(t)} |\nabla(\mathbf{U}-\mathbf{u})|^2(t)\, d\mathbf{x}\, dt \\
\le\ & E(0) + \varepsilon e^{-\alpha(T)} E(T) + \frac{\rho^{2m}}{\varepsilon} e^{-\alpha(T)} \|\mathbf{u}\|^2_{H^{m+1}}(T) \\
& + E(0) + \rho^{2m}\|\mathbf{u}\|^2_{H^{m+1}}(0) \\
& + \varepsilon \int_0^T \int_\Omega C_2 \left(1 + \|\mathbf{u}\|^2_{H^2}(t)\right) e^{-\alpha(t)} |\mathbf{U}-\mathbf{u}|^2\, d\mathbf{x}\, dt \\
& + \frac{C}{\varepsilon} \int_0^T \int_\Omega C_2 \left(1 + \|\mathbf{u}\|^2_{H^2}(t)\right) e^{-\alpha(t)} \left|\mathbf{u} - \mathcal{P}_\sigma \widehat{\mathcal{R}^m_{\rho,h}} \mathbf{u}\right|^2 d\mathbf{x}\, dt \\
& + C_4 \rho^{2m} \int_0^T e^{-\alpha(t)} \{\|\mathbf{u}_t\|^2_{H^{m+1}}(t) + \|\mathbf{u}\|^2_{H^{m+1}}(t) \left(1 + \|\mathbf{U}\|^2_{H^2}(t) + \|\mathbf{u}\|^2_{H^2}(t)\right)\}\, dt\,.
\end{aligned} \tag{60}
$$

But from the Poincáre inequality, we get

$$
\begin{aligned}
& \int_0^T C_2(1 + \|\mathbf{u}\|^2_{H^2}(t)) e^{-\alpha(t)} \int_\Omega |\mathbf{U}-\mathbf{u}|^2\, d\mathbf{x}\, dt \\
\le\ & C(|\Omega|^{\frac{2}{n}}) \int_0^T \{1 + \|\mathbf{u}\|^2_{H^2}(t)\} e^{-\alpha(t)} \int_\Omega |\nabla(\mathbf{U}-\mathbf{u})|^2\, d\mathbf{x}\, dt.
\end{aligned} \tag{61}
$$

Hence choosing a sufficiently small $\varepsilon > 0$, if $\|\mathbf{u}\|^2_{H^{m+1}}(t) < M$ for $0 \le t \le T$ and $\rho^m < \frac{C}{M}$, we have the following theorem of error estimates.

Theorem 6. *Let $\widehat{A^V}$ and A^P be the sets of the velocity shape functions and the pressure shape functions with m-th order consistency , respectively. Suppose $(\widehat{A^V}, A^P)$ is non-degenerate and satisfies the inf-sup condition. And assume $(\mathbf{u}, p) \in L^2(0,\infty : H^2_0(\Omega)) \cap L^\infty(0,\infty : H^1_0(\Omega)) \times L^2(0,\infty : H^1(\Omega)/\mathbb{R})$ is the solution of the Navier-Stokes equations (34) for $\mathbf{f} \in L^2(0,\infty : L^2(\Omega))$ and $(\mathbf{U}, P) \in C^1([0,T] : \mathcal{V}) \times C^0([0,T] : \mathcal{W})$ is the MLSRK solution of the discrete Navier-Stokes equation (35). Then there are some positive constants c, c_1 such that for $\rho^m < \frac{C}{M}$ where $\|\mathbf{u}\|^2_{H^{m+1}}(t) < M$, the following error estimates hold.*

$$
\begin{aligned}
& \|\mathbf{U}-\mathbf{u}\|^2_{L^2}(T) + \int_0^T e^{c_1(T-t)} \|\nabla(\mathbf{U}-\mathbf{u})\|^2_{L^2}(t)\, dt \\
\le\ & C\rho^{2m} e^{c_1 T} \Bigg[\|\mathbf{u}_0\|^2_{H^{m+1}} + \|\mathbf{u}\|^2_{H^{m+1}}(T) + \int_0^T e^{-c_1 t} \{\|\mathbf{u}_t\|^2_{H^{m+1}}(t) \\
& + \|\mathbf{u}\|^2_{H^{m+1}}(t) \left(1 + \|\mathbf{U}\|^2_{H^2}(t) + \|\mathbf{u}\|^2_{H^2}(t)\right)\}\, dt \Bigg]\,.
\end{aligned} \tag{62}
$$

4. **Numerical Examples.** In this section, we show numerical examples. We obtained the L^2-error estimates for the velocity for the non-stationary incompressible Stokes and Navier-Stokes equations, respectively, with zero boundary condition. In each case, the error is $O(\rho)$ for the fixed time T for both equations from the assumption of the regularity of the true solution. The emphasis in this section is on verifying these results numerically.

4.1. **Shape functions for velocity.** In this subsection, we show an example of test functions for velocity. The decay profile of shape functions on the boundary will be presented numerically. Consider $\Omega = [0,1]\times[0,1] \subset \mathbb{R}^2$, and the *regular* node set $\Lambda = \left\{ \left(\frac{i}{n}, \frac{j}{n}\right) \mid i = 0, \cdots, n \;\; j = 0, \cdots, n \right\}$, where n is some positive constant.

Let $\Phi = S(x)\,S(y)$,where S is defined as the following.

$$S(t) = \begin{cases} \frac{2}{3} - 4|t|^2(1-|t|) & \text{for} \quad |t| \le \frac{1}{2} \\ \frac{4}{3}(1-|t|)^3 & \text{for} \quad \frac{1}{2} < |t| < 1 \\ 0 & \text{for} \quad 1 \le |t| \;. \end{cases} \tag{63}$$

We notate the shape functions ϕ_i with window functions Φ. The transformed shape functions are defined by the following.

$$\widehat{\phi}_i(\mathbf{x}) = \begin{cases} \sum_j d_{(i,j)}\phi_j(\mathbf{x}) & \text{if} \;\; \mathrm{supp}(\phi_i) \cap \partial\Omega \ne \phi \\ \phi_i & \text{otherwise} \;, \end{cases}$$

where $[d_{(i,j)}] = [\phi_i(\mathbf{x}_j)]^{-1}$.

Nodes	h	γ	ρ	M
121	1.00e-01	1.8	1.8000e-01	1.719250e-02
441	5.00e-02	1.4	7.0000e-02	4.882729e-03
1681	2.50e-02	1.2	3.0000e-02	1.340959e-03
6561	1.25e-02	1.1	1.3750e-02	2.673020e-04
25921	6.25e-03	1.05	6.5625e-03	4.274686e-05

TABLE 1. $M = max_{\mathbf{x}\in\partial\Omega}\widehat{\phi}_i$

In table 1, h, γ and ρ stand for the nodal distance, a dilation parameter and the support radius of the shape function. The maximum value of shape functions on the boundary decreases as shown in the above. Since we are considering the first order shape functions, the expected decay ratio is 2. This numerical example shows bigger decay ratio than 2.

4.2. **Convergence of numerical solutions.** We consider the domain $\Omega = [0,1]\times[0,1]$ and take $(\mathbf{u}, p)$ as followings,

$$u = sin^2(\pi x) sin(\pi y) cos(\pi y) exp(-t), \tag{64}$$

$$v = -sin(\pi x) cos(\pi x) sin^2(\pi y) exp(-t), \tag{65}$$

$$p = (x^2 - y^2) exp(-t), \tag{66}$$

where $\mathbf{u} = (u, v)$. Next we calculate corresponding external forces $\mathbf{f}_S$ and $\mathbf{f}_{NS}$ of the following dimensionless form of the Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}_S \,, \qquad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \,, \tag{67}$$

where $\nu = 1$, and the Navier-Stokes equations respectively,

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}_{NS} \,, \qquad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \,, \tag{68}$$

where Re stands for the Reynolds number. Now we have the exact solutions of the Stokes equations and the Navier-Stokes equations for the above external forces

$\mathbf{f}_S$ and $\mathbf{f}_{NS}$ with corresponding boundary conditions. We want to compare relative errors between exact solutions and numerical solutions, for the Stokes equations and the Navier-Stokes equations respectively. In case of Navier-Stokes equations, we take $Re = 100$. With the above external force $\mathbf{f}_S$ and $\mathbf{f}_{NS}$ with boundary conditions from (64), (65) and (66), we compute the MLSRK solutions for the discrete Stokes equations (10) and the Navier-Stokes equations (35). Here we used the d'Alembert's principle to impose boundary conditions, we refer [6] for details. Node distributions for the velocity and the pressure are considered to satisfy the *inf-sup condition*, in fact, the velocity nodes are located at the points $(\frac{i}{2n}, \frac{j}{2n})$ for $i, j = 1, 2, \cdots, 2n$ and the pressure nodes are assigned at the points $(\frac{i}{n}, \frac{j}{n})$ for $i, j = 1, 2, \cdots, n$, as shown in figure 1. Since we assumed that $\mathbf{u} \in H^2(\Omega)$, we use the first order shape

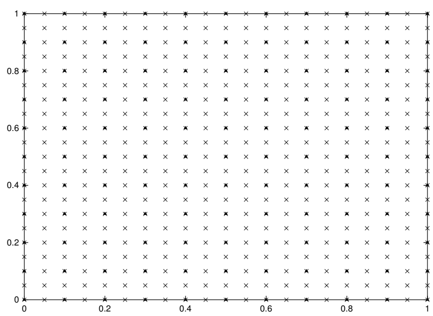


FIGURE 1. Distribution of Nodes(Velocity and Pressure)

functions, i.e. $m = 1$. We chose shape functions of velocity and pressure with dilation parameters ρ_v and ρ_p such that $\rho_v = 1.1 \times h$ and $\rho_p = 2\rho_v$, while the window function is the product of cubic spline functions defined in (63). This means that we take the support size of the pressure shape function to be double of the velocity shape function. This type of shape functions seems to satisfy the *inf-sup condition.* The shape functions of the velocity and the pressure associated with the window function $\Phi(x, y)$ are drawn in Fig. 2. For the discretization in

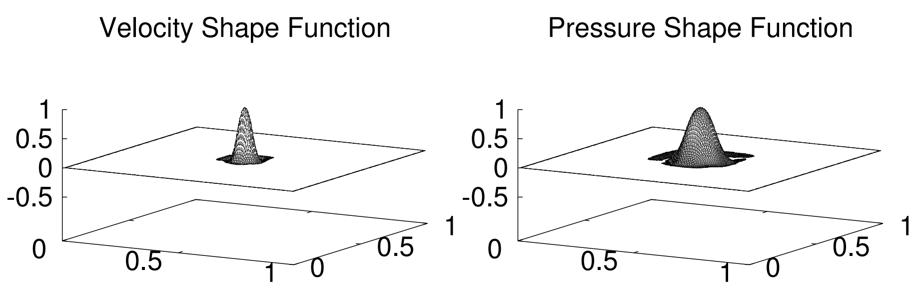


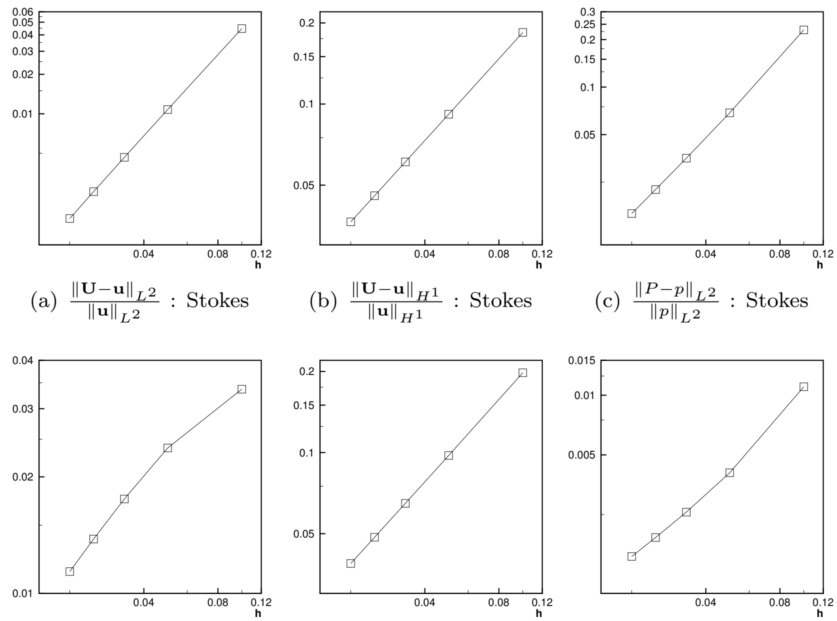
FIGURE 2. Velocity Shape Function(left) and Pressure Shape Function(right)

time, we use the backward Euler method which is an implicit scheme. At each time step, the Navier-Stokes equations are linearized by the following.

$$\mathbf{u}_n^k - \mathbf{u}_{n-1} - \Delta t \left(\nu \Delta \mathbf{u}_n^k + (\mathbf{u}_{n-1} \cdot \nabla)\mathbf{u}_n^k + \nabla p\right) = \Delta t \mathbf{f}$$

Here, $\mathbf{u}_{n-1}$ is the solution of $(n-1)$-th time step, and $\mathbf{u}_n^k$ is the solution of inner iteration. The solution of n-th step is obtained when the successive error of the inner iteration is small enough. The linear system consists of the discrete momentum equation and the discrete continuity equation, and it is a kind of saddle point problem. We used the bi-conjugate gradient method as a solver. For the numerical integration, we used two by two Gaussian-quadrature on the square back-cell.

As shown in tables 2,3 and 4 for the Stokes equations and in tables 5, 6 and 7 for the Navier-Stokes equations, the L^2-errors for the velocity and the pressure coincide with our analysis. Furthermore, we can view the uniform boundedness of these errors in time. The L^2-error plots for the velocity and pressure at time $t = 1.0$ are shown in Fig. 3 as a reference.



(a) $\frac{\|\mathbf{U}-\mathbf{u}\|_{L^2}}{\|\mathbf{u}\|_{L^2}}$: Stokes (b) $\frac{\|\mathbf{U}-\mathbf{u}\|_{H^1}}{\|\mathbf{u}\|_{H^1}}$: Stokes (c) $\frac{\|P-p\|_{L^2}}{\|p\|_{L^2}}$: Stokes

(d) $\frac{\|\mathbf{U}-\mathbf{u}\|_{L^2}}{\|\mathbf{u}\|_{L^2}}$: Navier-Stokes (e) $\frac{\|\mathbf{U}-\mathbf{u}\|_{H^1}}{\|\mathbf{u}\|_{H^1}}$: Navier-Stokes (f) $\frac{\|P-p\|_{L^2}}{\|p\|_{L^2}}$: Navier-Stokes

FIGURE 3. Decay rates of relative errors at time $t = 1.0$

Nodes	Stokes equations			
Vel. × Pres.	t=0.01	t=0.1	t=0.5	t=1.0
121 × 36	1.98907220e-02	3.99434253e-02	4.46688766e-02	4.46728916e-02
441 × 121	4.52804333e-03	1.01725811e-02	1.07787172e-02	1.07787461e-02
961 × 256	1.93789420e-03	4.48012356e-03	4.65470350e-03	4.65470487e-03
1681 × 441	1.06246032e-03	2.47702703e-03	2.54934122e-03	2.54934138e-03
2601 × 676	6.64661575e-04	1.55276747e-03	1.58960783e-03	1.58960787e-03

TABLE 2. Relative L^2- errors of $\mathbf{U}$ $\left(\frac{\|\mathbf{U}-\mathbf{u}\|_{L^2}}{\|\mathbf{u}\|_{L^2}}\right)$ for the Stokes equations.

Nodes Vel. × Pres.	Stokes equations			
	t=0.01	t=0.1	t=0.5	t=1.0
121 × 36	1.85910324e-01	1.84439007e-01	1.84491401e-01	1.84491503e-01
441 × 121	9.18278031e-02	9.16429909e-02	9.16467639e-02	9.16467642e-02
961 × 256	6.10325832e-02	6.09773132e-02	6.09778591e-02	6.09778592e-02
1681 × 441	4.57122666e-02	4.56885780e-02	4.56886702e-02	4.56886702e-02
2601 × 676	3.65417798e-02	3.65294102e-02	3.65294126e-02	3.65294126e-02

TABLE 3. Relative H^1- errors of $\mathbf{U}$ $(\frac{\|\mathbf{U}-\mathbf{u}\|_{H^1}}{\|\mathbf{u}\|_{H^1}})$ for the Stokes equations.

Nodes Vel. × Pres.	Stokes equations			
	t=0.01	t=0.1	t=0.5	t=1.0
121 × 36	2.73942903e-01	2.37766161e-01	2.30324142e-01	2.30317872e-01
441 × 121	7.60134539e-02	6.93491934e-02	6.86912738e-02	6.86912425e-02
961 × 256	3.80363521e-02	3.56011420e-02	3.54455531e-02	3.54455518e-02
1681 × 441	2.36609598e-02	2.24827693e-02	2.24265704e-02	2.24265703e-02
2601 × 676	1.64912408e-02	1.58257902e-02	1.58000068e-02	1.58000068e-02

TABLE 4. Relative L^2- errors of P $(\frac{\|P-p\|_{L^2}}{\|p\|_{L^2}})$ for the Stokes equations.

Nodes Vel. × Pres.	Navier-Stokes equations($Re = 100$)			
	t=0.01	t=0.1	t=0.5	t=1.0
121 × 36	1.16159163e-02	1.18839566e-02	1.82552230e-02	3.36833140e-02
441 × 121	2.29975273e-03	2.82557690e-03	1.04839345e-02	2.37514134e-02
961 × 256	9.33575207e-04	1.52808990e-03	7.65000365e-03	1.75373665e-02
1681 × 441	5.01027145e-04	1.09031059e-03	6.02301109e-03	1.38181808e-02
2601 × 676	3.12274611e-04	8.71759828e-04	4.96910134e-03	1.13874765e-02

TABLE 5. Relative L^2- errors of $\mathbf{U}$ $(\frac{\|\mathbf{U}-\mathbf{u}\|_{L^2}}{\|\mathbf{u}\|_{L^2}})$ for the Navier-Stokes equations.

Nodes Vel. × Pres.	Navier-Stokes equations($Re = 100$)			
	t=0.01	t=0.1	t=0.5	t=1.0
121 × 36	1.89041294e-01	1.89161988e-01	1.91995761e-01	1.97861163e-01
441 × 121	9.22401536e-02	9.23304300e-02	9.37851805e-02	9.75519405e-02
961 × 256	6.11540663e-02	6.12214862e-02	6.21251236e-02	6.47707073e-02
1681 × 441	4.57620074e-02	4.58120646e-02	4.64570953e-02	4.84889811e-02
2601 × 676	3.65665149e-02	3.66046394e-02	3.71038617e-02	3.87532709e-02

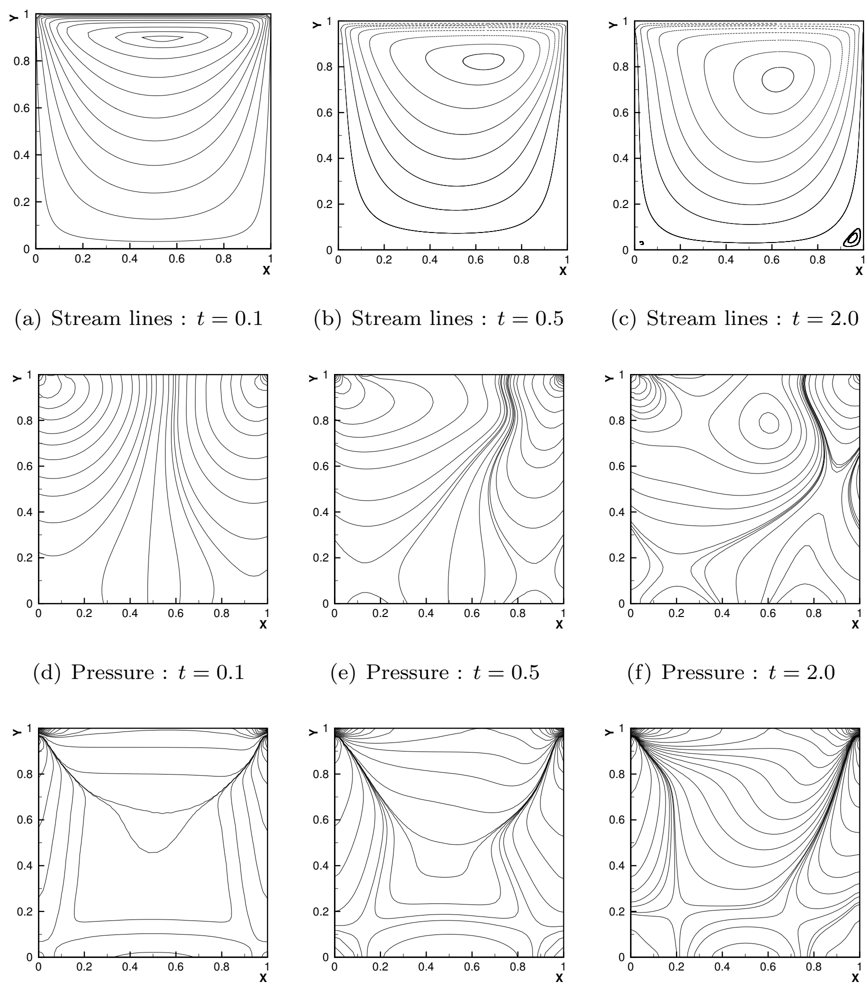
TABLE 6. Relative H^1- errors of $\mathbf{U}$ $(\frac{\|\mathbf{U}-\mathbf{u}\|_{H^1}}{\|\mathbf{u}\|_{H^1}})$ for the Navier-Stokes equations.

4.3. **The lid driven cavity flow.** Though we studied problems with the homogeneous boundary condition for the velocity, the driven cavity flow is calculated as an example of problems with the non-homogeneous boundary condition. In figure 4, the streamlines and the contour lines of the pressure and the vorticity are depicted

Nodes	Navier-Stokes equations($Re = 100$)			
Vel. $\times$ Pres.	t=0.01	t=0.1	t=0.5	t=1.0
121 $\times$ 36	1.02315848e-02	1.02625024e-02	1.06547697e-02	1.10323413e-02
441 $\times$ 121	2.59315599e-03	2.63763555e-03	3.41949068e-03	4.06204224e-03
961 $\times$ 256	1.16957475e-03	1.23078139e-03	2.01078538e-03	2.57137796e-03
1681 $\times$ 441	6.66906869e-04	7.37124165e-04	1.44739887e-03	1.91481713e-03
2601 $\times$ 676	4.32326722e-04	5.07244332e-04	1.14198542e-03	1.53618512e-03

TABLE 7. Relative L^2- errors of P $(\frac{\|P-p\|_{L^2}}{\|p\|_{L^2}})$ for the Navier-Stokes equations.

at time $t = 0.1,\ 0.5,\ 2.0$ with $Re = 100$. The profiles at time $t = 2.0$ coincide with the solution of the stationary case(Figure 4 in [3]).



(a) Stream lines : $t = 0.1$ (b) Stream lines : $t = 0.5$ (c) Stream lines : $t = 2.0$

(d) Pressure : $t = 0.1$ (e) Pressure : $t = 0.5$ (f) Pressure : $t = 2.0$

(g) Vorticity : $t = 0.1$ (h) Vorticity : $t = 0.5$ (i) Vorticity : $t = 2.0$

FIGURE 4. Navier-Stokes($Re = 100$) driven cavity flows

5. **Conclusions.** The meshfree method for the non-stationary incompressible Stokes and the Navier-Stokes problems are analyzed mathematically, and several numerical examples are implemented successfully. In this paper, we have obtained the solvability of the discrete Stokes equations and the Navier-Stokes equations by the meshfree method. And we have shown the convergence of the numerical solutions to the true solutions for each case, which result in the L^2-error estimate of the velocity. The numerical results show good agreement with the theoretic analysis for the convergence of the discrete solutions.

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