

## MESHLESS METHOD FOR THE STATIONARY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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**ABSTRACT.** Mathematical analysis is achieved on a meshless method for the stationary incompressible Stokes and Navier-Stokes equations. In particular, the Moving Least Square Reproducing Kernel(MLSRK) method is employed. The existence of discrete solution and its error estimate are obtained. As a numerical example for convergence analysis, we compute the numerical solutions for these equations to compare with exact solutions. Also we solve the driven cavity flow numerically as a test problem.

**1. Introduction.** The objective of this paper is to develop the numerical theory for the Galerkin formulation using the MLSRK(moving least square reproducing kernel) method especially for the stationary incompressible Stokes and Navier-Stokes equations.

Several methods for meshless approximations were proposed for various applications. We note that Smoothed Particle Hydrodynamics(SPH) by Gingold and Monaghan(1977)[1], Reproducing Kernel Particle Method(RKPM) by Liu et al.(1995)[6, 5], Diffuse Element Method(DEM) by Nayroles et al.(1992)[11], Element Free Galerkin Method(EFG) by Belytschko et al.(1994)[9] and Partition of Unity Finite Element Method(PUFEM) by Babuška and Melenk(1995)[10] were proposed. In particular, we are interested in the applications of Moving Least Square Reproducing Kernel Galerkin Method(MLSRK) proposed by Liu et al.(1996)[8] to the incompressible Navier-Stokes equations. One distinct advantage of this

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method over the standard finite element method is that it requires simple distribution of nodes, not the complex mesh generation depend on the geometry of the flow domain. Another advantage is that the desired regularity of the approximate solution can be readily achieved by introducing suitable window function with sufficient regularity.

Though there has been keen interest in developing meshless approximations for the Galerkin formulation of the partial differential equations in engineering, mathematical analysis on the existence and convergence criterion of discrete solution has not been made yet as far as we have known. In this paper, We have obtained the solvability and the convergence for successive approximation of solution of the Stokes and the Navier-Stokes equations, which results in the  $H^1$ -error estimate of the velocity.

As a numerical example, we calculate the numerical solutions for the Stokes and the Navier-Stokes equations in two dimension and the errors of each components of the numerical solution are tabulated. Also the driven cavity flow is calculated numerically as a test case with non-zero boundary condition and the several plots for the numerical solution are shown in this paper.

## 2. Moving Least Square Reproducing Kernel Method.

### 2.1. Reproducing Formula.

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\partial\Omega$  and  $u(x)$  be a smooth function defined in  $\Omega$ . Define the set of all basis polynomials of order less than or equal to  $m$

$$\mathcal{P}_m(x) = \{P_\alpha(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid |\alpha| = \alpha_1 + \cdots + \alpha_n \leq m\}.$$

Here, the number of all components in  $\mathcal{P}_m(x)$  is  $\frac{(n+m)!}{n!m!}$ . We choose a smooth non-negative window function  $\Phi(x)$  which has a compact support, say,  $\text{supp } \Phi \subset B_1(0)$ . To describe the MLSRK approximation of  $u(x)$  with  $m$ -th order consistency, let us introduce a localized error residual functional

$$J(\mathbf{a}(\bar{x})) \equiv \int_{\Omega} \left| u(x) - \mathcal{P}_m \left( \frac{x - \bar{x}}{\rho} \right) \cdot \mathbf{a}(\bar{x}) \right|^2 \Phi_\rho(x - \bar{x}) dx,$$

where  $\Phi_\rho(x - \bar{x}) = \frac{1}{\rho^n} \Phi \left( \frac{x - \bar{x}}{\rho} \right)$  and  $\rho > 0$  is a dilation parameter. Minimizing the quadratic functional  $J(\mathbf{a}(\bar{x}))$ , we find that the minimizer  $\mathbf{a}(\bar{x})$  satisfies

$$\mathbf{a}(\bar{x}) = M^{-1}(\bar{x}) \int_{\Omega} \mathcal{P}_m^T \left( \frac{x - \bar{x}}{\rho} \right) u(x) \Phi_\rho(x - \bar{x}) dx,$$

here the matrix  $M(\bar{x})$  is so called the moment matrix defined by

$$M(\bar{x}) \equiv \int_{\Omega} \mathcal{P}_m^T \left( \frac{x - \bar{x}}{\rho} \right) \mathcal{P}_m \left( \frac{x - \bar{x}}{\rho} \right) \Phi_\rho(x - \bar{x}) dx. \quad (1)$$

Since the polynomial basis  $\mathcal{P}_\alpha(x)$ 's are linearly independent,  $M(\bar{x})$  is always invertible and  $\det M(\bar{x}) > 0$ . Now the local approximation of  $u(x)$  near  $\bar{x}$  is obtained as the following :

$$\begin{aligned} U(x, \bar{x}) &\equiv \mathcal{P}_m \left( \frac{x - \bar{x}}{\rho} \right) \cdot \mathbf{a}(\bar{x}) \\ &= \mathcal{P}_m \left( \frac{x - \bar{x}}{\rho} \right) M^{-1}(\bar{x}) \int_{\Omega} \mathcal{P}_m^T \left( \frac{y - \bar{x}}{\rho} \right) u(y) \Phi_\rho(y - \bar{x}) dy. \end{aligned}$$

For fixed  $\bar{x} \in \Omega$ , the manipulation is the standard weighted least square procedure. Since  $\bar{x}$  is an arbitrary point in  $\Omega$  for the weight least square procedure, we may choose  $\bar{x} = x$  and we obtain a global approximation of  $u(x)$ . More precisely the global approximation operator is defined by

$$\begin{aligned} G u(x) &\equiv U(x, x) \\ &= \mathcal{P}_m(0)M^{-1}(x) \int_{\Omega} \mathcal{P}_m^T \left( \frac{y-x}{\rho} \right) u(y) \Phi_{\rho}(y-x) dy. \end{aligned} \tag{2}$$

This formulation is so called reproducing kernel formulation by Liu et al.(1995). For simplicity, we define the correction function as

$$C(\rho, y-x, x) = \mathcal{P}_m(0)M^{-1}(x) \mathcal{P}_m^T \left( \frac{y-x}{\rho} \right)$$

and the kernel function as

$$K_{\rho}(y-x, x) \equiv C(\rho, y-x, x) \Phi_{\rho}(y-x),$$

then the global approximation (2) is written in a convolution form

$$G u(x) = \int_{\Omega} K_{\rho}(y-x, x) u(y) dy. \tag{3}$$

For this global approximation, any polynomial of order less than or equal to  $m$  satisfies

$$G u(x) = u(x), \tag{4}$$

and we call this property as  $m$ -th order consistency.

To show  $m$ -th order consistency for the above MLSRK approximation, let  $u(y)$  be a polynomial of order less than or equal to  $m$ . Then it is represented by the form

$$u(y) = \sum_{|\beta| \leq m} c_{\beta}(x) \left( \frac{y-x}{\rho} \right)^{\beta},$$

where  $\beta = (\beta_1, \dots, \beta_n)$  is a multi-index and  $x^{\beta} = x_1^{\beta_1} \dots x_n^{\beta_n}$  for  $x \in \mathbb{R}^n$ . Here we note that the first coefficient  $c_{\beta_0}(x)$  with  $\beta_0 = (0, \dots, 0)$  is the polynomial  $u(x)$ , i.e.,

$$c_{\beta_0}(x) = u(x). \tag{5}$$

From the definition of moment matrix (1) and the reproducing kernel approximation (3) we have

$$\begin{aligned} G u(x) &= \mathcal{P}_m(0)M^{-1}(x) \int_{\Omega} \mathcal{P}_m^T \left( \frac{y-x}{\rho} \right) \sum_{|\beta| \leq m} c_{\beta}(x) \left( \frac{y-x}{\rho} \right)^{\beta} \Phi_{\rho}(y-x) dy \\ &= \mathcal{P}_m(0) \sum_{|\beta| \leq m} c_{\beta}(x) \mathbf{e}_{\beta} = c_{\beta_0}(x), \end{aligned}$$

where  $\mathbf{e}_{\beta} = [0, \dots, 1, \dots, 0]^T$  is the  $\beta$ -th standard basis of dimension  $\frac{(n+m)!}{n!m!}$ . Therefore, from the identity (5), we have the  $m$ -th order consistency of (4).

Now we define the shape function to develop the theory of MLSRK for the Stokes and the Navier-Stokes equations. For a given set of nodes  $\Lambda = \{x_i | i = 1, \dots, NP\}$ ,

employing discretized moment matrix

$$M^h(x) = \sum_{i=1}^{NP} \mathcal{P}^T \left( \frac{x - x_i}{\rho_i} \right) \mathcal{P} \left( \frac{x - x_i}{\rho_i} \right) \Phi_{\rho_i}(x - x_i), \quad (6)$$

we define discretized kernel function

$$\begin{aligned} K_{\rho_i}^h(x - x_i, x) &= C^h(\rho_i, x - x_i, x) \Phi_{\rho_i}(x - x_i) \\ &= \mathcal{P}(0)(M^h)^{-1}(x) \mathcal{P}^T \left( \frac{x - x_i}{\rho_i} \right) \Phi_{\rho_i}(x - x_i). \end{aligned} \quad (7)$$

This set of functions will be used in this paper as MLSRK shape functions, which is called simply shape functions if there is no confusion. Also we will denote briefly  $K_{\rho_i}^h(x - x_i, x)$  as  $\phi_i(x)$  for the window function  $\Phi$ .

## 2.2. Node Distribution.

In the finite element method, the mesh generation follows *quasi-uniform* or *regular* condition. Similarly, we have the following condition. Node set is not concentrated in some region of domain, and the support of each shape function overlap sufficiently many times with other shape function's support. The overlapping condition ensures the invertibility of the moment matrix defined in (6). In detail, we want each point  $x \in \Omega$  is contained in the supports of at most  $L$  shape functions where  $L$  is some fixed positive integer independent of the number of nodes, i.e., for the support of shape function, the maximum number of intersection is bounded independent of the number of nodes. In this manner we define the followings.

DEFINITION 2.1. Let  $\Lambda = \{x_i | i = 1, \dots, NP\}$  be the set of nodes. We define  $\Lambda$  be a regular node set if the followings hold.

i) There exist  $C_1 > 0$  independent of  $NP$  such that

$$\min_i h_i \geq C_1 \max_i h_i$$

where  $h_i = \min_{j \neq i} |x_i - x_j|$ . So there is a characteristic distance  $h$  such that  $h \leq h_i \leq C_2 h$  for all  $i$ .

ii) Let  $\rho_i = \gamma h_i$ , for some fixed  $\gamma > 1$ . There exist  $C_\gamma > 0$  depend only on  $\gamma$  such that

$$\min_i N(i, \gamma) \geq C_\gamma \max_i N(i, \gamma)$$

where  $N(i, \gamma)$  be the number of nodes contained in  $B_{\rho_i}(x_i)$ . We also let  $\rho = \gamma h$  be characteristic radius.

REMARK 1. Let us call  $\gamma$  in Definition 2.1 as dilation ratio for regular node set. The parameter  $\rho_i$  will be used as dilation parameter of shape function  $\phi_i(x)$  so that the support of  $\phi_i(x)$  is contained in  $B_{\rho_i}(x_i)$ . It is not hard, but rather complicated, to check  $\gamma > m$  to ensure invertibility of moment matrix while generating shape functions with  $m$ -th order consistency.

DEFINITION 2.2. Let  $A = \{\phi_i | i = 1, \dots, NP\}$  be the set of MLSRK shape functions generated by the window function  $\Phi$  for the regular node set  $\Lambda = \{x_i | i = 1, \dots, NP\}$ , with dilation parameter  $\rho_i$  as defined in the Definition 2.1. Then the shape function set  $A$  is admissible if there is a positive constant  $\beta_0$  independent of  $\rho$  such that

$$\beta_0^{-1} \rho^n \|a\|^2 \geq \sum_{\alpha=1}^n \sum_{i,j=1}^{NP} \int_{\Omega} \phi_i \phi_j dx a_i^\alpha a_j^\alpha \geq \beta_0 \rho^n \|a\|^2, \quad (8)$$

for all  $a^\alpha \in \mathbb{R}^{NP}$ ,  $\alpha = 1, \dots, n$ .

Note that the above *regular* and *admissibility* conditions imply certain uniform condition for the node distance and support radius of shape functions. In making shape function, dilation parameter  $\rho_i$  and node distance  $h_i$  are depend on each  $x_i$ , but with assuming *regular* node distribution, we may consider  $\rho_i$  and  $h_i$  as independent parameter with respect to each  $x_i$ .

**2.3. Projection Error Estimate.**

For the convergence analysis, we need interpolation error estimate between the solution space and the projection generated by the set of shape functions. We define discrete projection and find projection error estimate.

DEFINITION 2.3. Let  $A = \{\phi_i | i = 1, \dots, NP\}$  be the admissible set of MLSRK shape functions generated by the window function  $\Phi$  for the regular node set  $\Lambda = \{x_i | i = 1, \dots, NP\}$ . Let  $u(x) \in C^0(\Omega)$  be a function and  $\rho_i > 0$  is a dilation parameter defined in the Definition 2.1. We define the discrete projection as

$$\mathcal{R}_{\rho,h}^m u(x) \equiv \sum_{i=1}^{NP} u(x_i) \phi_i(x) = \sum_{x_i \in \Lambda(x)} u(x_i) \phi_i(x),$$

where  $\phi_i(x) = K_{\rho_i}^h(x - x_i, x)$  as in (7) and  $\Lambda(x) = \{x_i \in \Lambda | x \in \text{supp}(\phi_i(x)) \cap \bar{\Omega}\}$ . Here,  $m$  denotes the order of generating polynomial basis  $\mathcal{P}_m$ ,  $\rho$  is the characteristic dilation parameter and  $h$  stands for the distance between the nodes.

The following theorem implies that the projection error converges to zero as we enlarge the node set holding *regular* condition.

THEOREM 2.1. Assume the window function  $\Phi(x) \in C_0^m(\mathbb{R}^n)$  and  $v(x) \in C^{m+1}(\bar{\Omega})$ , where  $\Omega$  is a bounded open set in  $\mathbb{R}^n$ . Let  $\Lambda = \{x_i | i = 1, \dots, NP\}$  be a regular node set and  $A = \{\phi_i | i = 1, \dots, NP\}$  be the set of admissible shape functions. Suppose the boundary of  $\Omega$  is smooth and  $\text{supp}\phi_i \cap \bar{\Omega}$  is convex for each  $i$ . If  $m$  and  $p$  satisfy

$$m > \frac{n}{p} - 1 \quad ,$$

then the following interpolation estimate holds

$$\|v - \mathcal{R}_{\rho,h}^m v\|_{W^{k,p}(\Omega)} \leq C_k \rho^{m+1-k} \|v\|_{W^{m+1,p}(\Omega)}, \quad \text{for all } 0 \leq k \leq m \quad , \quad (9)$$

here  $\rho$  stands for the characteristic dilation parameter, i.e., some positive number satisfying  $\min_i \rho_i \leq \rho \leq \max_i \rho_i$ .

*Proof.* Let  $v(x) \in C^{m+1}(\bar{\Omega})$  be given. By definition of projection, we have

$$\mathcal{R}_{\rho,h}^m v(x) = \sum_{x_i \in \Lambda(x)} v(x_i) \phi_i(x)$$

and taking derivatives yields, for  $|\beta| \leq m$ ,

$$D_x^\beta \mathcal{R}_{\rho,h}^m v(x) = \sum_{x_i \in \Lambda(x)} v(x_i) D_x^\beta \phi_i(x).$$

Evaluating the value  $v(x_i)$  by Taylor expansion of  $v$  at  $x$ , we have

$$\begin{aligned} v(x_i) &= \sum_{|\alpha| \leq m} \frac{1}{\alpha!} (x_i - x)^\alpha D_x^\alpha v(x) \\ &\quad + \sum_{|\alpha|=m+1} c(\alpha) \int_0^1 (1-\theta)^m D_x^\alpha v(x + \theta(x_i - x)) d\theta (x_i - x)^\alpha. \end{aligned}$$

Using the above expansion and the reproducing property of shape functions, we have the identity

$$\begin{aligned} D_x^\beta \mathcal{R}_{\rho,h}^m v(x) &= \sum_{x_i \in \Lambda(x)} \left[ \sum_{|\alpha| \leq m} \frac{1}{\alpha!} (x_i - x)^\alpha D_x^\alpha v(x) \right. \\ &\quad \left. + \sum_{|\alpha|=m+1} c(\alpha) \int_0^1 (1-\theta)^m D_x^\alpha v(x + \theta(x_i - x)) d\theta (x_i - x)^\alpha \right] D_x^\beta \phi_i(x) \\ &= \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D_x^\alpha v(x) \alpha! \delta_{\alpha\beta} \\ &\quad + \sum_{x_i \in \Lambda(x)} \left[ \sum_{|\alpha|=m+1} c(\alpha) \int_0^1 (1-\theta)^m D_x^\alpha v(x + \theta(x_i - x)) d\theta \right] (x_i - x)^\alpha D_x^\beta \phi_i(x). \end{aligned}$$

Therefore we have

$$\begin{aligned} &|D_x^\beta v(x) - D_x^\beta \mathcal{R}_{\rho,h}^m v(x)| \\ &\leq c(m) \sum_{x_i \in \Lambda(x)} \left[ \sum_{|\alpha|=m+1} \int_0^1 (1-\theta)^m |D_x^\alpha v(x + \theta(x_i - x))| d\theta \right] |x_i - x|^{m+1} |D_x^\beta \phi_i(x)|. \end{aligned}$$

Considering scaling, we find

$$|D_x^\beta \phi_i(x)| \leq c\rho^{-|\beta|},$$

where  $c = \max_i \sup_z |D_x^\beta K_1^h(z - x_i, z)|$ , note that  $K_{\rho_i}^h(x - x_i, x) = \phi_i(x)$ . Also if  $x \in \text{supp } \phi_i(x)$ , then  $|x_i - x| \leq c\rho$ .

Hence we find the following inequalities using the Minkowski inequality

$$\begin{aligned} &\int_\Omega |D_x^\beta v(x) - D_x^\beta \mathcal{R}_{\rho,h}^m v(x)|^p dx \\ &\leq c \int_\Omega \left[ \sum_{x_i \in \Lambda(x)} \sum_{|\alpha|=m+1} \int_0^1 (1-\theta)^m |D_x^\alpha v(x + \theta(x_i - x))| d\theta |x_i - x|^{m+1} |D_x^\beta \phi_i(x)| \right]^p dx \\ &\leq c\rho^{p(m+1-|\beta|)} \sum_{|\alpha|=m+1} \int_\Omega \left[ \int_0^1 (1-\theta)^m \sum_{x_i \in \Lambda(x)} |D_x^\alpha v(x + \theta(x_i - x))| \chi_{\text{supp}(\phi_i(x))} d\theta \right]^p dx \\ &\leq c\rho^{p(m+1-|\beta|)} \sum_{|\alpha|=m+1} \left[ \int_0^1 \left( \int_\Omega (1-\theta)^{mp} \sum_{x_i \in \Lambda(x)} |D_x^\alpha v(x + \theta(x_i - x))| \chi_{\text{supp}(\phi_i(x))} |^p dx \right)^{\frac{1}{p}} d\theta \right]^p. \end{aligned}$$

Now we let  $y = x + \theta(x_i - x)$ , then

$$\int_{\Omega} |D_x^\alpha v(x + \theta(x_i - x)) \chi_{\text{supp}(\phi_i(x))}|^p dx \leq \int_{B_{c(1-\theta)\rho}(x_i)} |D_x^\alpha v(y)|^p \frac{dy}{(1-\theta)^n}.$$

So we have

$$\begin{aligned} & \sum_{|\alpha|=m+1} \left[ \int_0^1 \left\{ \int_{\Omega} (1-\theta)^{mp} \left( \sum_{x_i \in \Lambda(x)} |D_x^\alpha v(x + \theta(x_i - x))| \chi_{\text{supp}(\phi_i(x))} \right)^p dx \right\}^{\frac{1}{p}} d\theta \right]^p \\ & \leq cL \sum_{|\alpha|=m+1} \left[ \int_0^1 (1-\theta)^{m-\frac{n}{p}} \left( \int_{\Omega} |D_x^\alpha v(y)|^p dy \right)^{\frac{1}{p}} d\theta \right]^p, \end{aligned}$$

here  $L = \max_i N(i, \rho)$  as in Definition 1. Therefore if  $m > \frac{n}{p} - 1$ , then

$$\begin{aligned} & \sum_{|\alpha|=m+1} \left[ \int_0^1 \left\{ \int_{\Omega} (1-\theta)^{mp} \left( \sum_{x_i \in \Lambda(x)} |D_x^\alpha v(x + \theta(x_i - x))| \chi_{\text{supp}(\phi_i(x))} \right)^p dx \right\}^{\frac{1}{p}} d\theta \right]^p \\ & \leq c \sum_{|\alpha|=m+1} \|D_x^\alpha v\|_{L^p}^p. \end{aligned}$$

Therefore we have, for  $|\beta| = k \leq m$ ,

$$\int_{\Omega} |D_x^\beta v(x) - D_x^\beta \mathcal{R}_{\rho,h}^m v(x)|^p dx \leq c \rho^{p(m+1-k)} \|v\|_{W^{m+1,p}}^p$$

if  $m > \frac{n}{p} - 1$ . □

For the discrete projection, we find the following Sobolev type embedding theorem which will be used for the analysis of the Navier-Stokes equations.

**THEOREM 2.2.** *We assume  $\text{supp} \phi_i \cap \Omega$  is convex. Suppose  $v \in C_0^1(\Omega)$  and the window function  $\Phi \in C_0^m(\mathbb{R}^n)$ ,  $m \geq 1$ . If  $p > n$  holds, then we have the following inequalities*

$$\|\nabla \mathcal{R}_{\rho,h}^m v\|_{L^p} \leq C \|\nabla v\|_{L^p}, \tag{10}$$

$$\sup_{\Omega} |\mathcal{R}_{\rho,h}^m v| \leq C |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|\nabla v\|_{L^p}. \tag{11}$$

*Proof.* Let  $\rho$  be given and  $\Lambda = \{x_i | i = 1, \dots, NP\}$  be the regular node set. By the definition of the discrete projection  $\mathcal{R}_{\rho,h}^m$ , we have

$$\mathcal{R}_{\rho,h}^m v(x) = \sum_{x_i \in \Lambda} v(x_i) \phi_i(x)$$

and

$$\frac{\partial}{\partial x_\alpha} \mathcal{R}_{\rho,h}^m v(x) = \sum_{x_i \in \Lambda} v(x_i) \frac{\partial}{\partial x_\alpha} \phi_i(x)$$

where  $x \in \Omega$  and  $x_\alpha, \alpha = 1, \dots, n$  is component of  $x$ . By Taylor series expansion of the function  $v(x)$  for given  $y$ , we can write

$$v(x_i) = v(y) + \int_0^1 (x_i - y) \cdot \nabla_y v(y + \theta(x_i - y)) d\theta.$$

Hence we obtain

$$\begin{aligned} \frac{\partial}{\partial x_\alpha} \mathcal{R}_{\rho,h}^m v(x) &= \sum_{x_i \in \Lambda} v(x) \frac{\partial}{\partial x_\alpha} \phi_i(x) \\ &\quad + \sum_{x_i \in \Lambda} \int_0^1 (x_i - x) \cdot \nabla_y v(x + \theta(x_i - x)) d\theta \frac{\partial}{\partial x_\alpha} \phi_i(x). \end{aligned}$$

From the linear consistency of the shape functions, the first summation of the above equation vanishes, i.e.,

$$\sum_{x_i \in \Lambda} v(x) \frac{\partial}{\partial x_\alpha} \phi_i(x) = 0.$$

Also our construction of the shape function  $\phi_i(x)$  implies

$$|x_i - x| |\nabla_x \phi_i(x)| \leq C$$

and each  $x$  is contained in the finite number of supports of  $\phi_i(x)$ 's from the overlapping assumption of the shape functions. We set  $\Lambda(x) = \{x_i \in \Lambda \mid x \in \text{support of } \phi_i(x)\}$  and the number of the element of  $\Lambda(x)$  is bounded by for some fixed number  $L$ . Also we note that  $\text{support of } \phi_i(x) \subset B_{c\rho}(x_i)$  for some constant  $c$  independent of  $\rho$ . Thus it follows that

$$\begin{aligned} &\sum_{x_i \in \Lambda} \int_0^1 |x_i - x| |\nabla_x v(x + \theta(x_i - x))| d\theta \left| \frac{\partial}{\partial x_\alpha} \phi_i(x) \right| \\ &\leq C \sum_{x_i \in \Lambda(x)} \chi_{B_{c\rho}(x_i)} \int_0^1 |\nabla v(x + \theta(x_i - x))| d\theta. \end{aligned}$$

Therefore we have the following estimates, from the Hölder inequality and the Minkowski inequality,

$$\begin{aligned} \|\nabla \mathcal{R}_{\rho,h}^m v\|_{L^p} &\leq C \left[ \int_\Omega \left[ \sum_{x_i \in \Lambda(x)} \chi_{B_{c\rho}(x_i)} \int_0^1 |\nabla v(x + \theta(x_i - x))| d\theta \right]^p dx \right]^{\frac{1}{p}} \quad (12) \\ &\leq C \int_0^1 d\theta \left[ \int_\Omega \sum_{x_i \in \Lambda(x)} \chi_{B_{c\rho}(x_i)} |\nabla v(x + \theta(x_i - x))|^p dx \right]^{\frac{1}{p}} \\ &\leq C \int_0^1 (1 - \theta)^{-\frac{n}{p}} d\theta \left[ \int_\Omega |\nabla v(y)|^p dy \right]^{\frac{1}{p}} \leq CL \|\nabla v\|_{L^p} \end{aligned}$$

Note that the last inequality in (12), the finite intersection property of shape functions is crucial. Also remember our assumption  $p > n$ .

For the  $L^\infty$ -norm estimate, we have

$$\sup_\Omega |\mathcal{R}_{\rho,h}^m v| \leq C |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|\nabla \mathcal{R}_{\rho,h}^m v\|_{L^p} \leq C |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|\nabla v\|_{L^p}.$$

The first part of the above inequalities results from the usual Sobolev embedding and the second part have been proved at (12). Therefore we have proved our theorem.  $\square$



**2.4. Test Function Space and Boundary Transformation.**

To develop the theory using the Galerkin formulation based on MLSRK method, it is necessary to clarify the test function space. In this subsection we define the test function space denoted by  $V_0^h$  and describe how to construct this space. Also we prove the projection error estimate for this space.

Let  $\Lambda = \{x_i | i = 1, \dots, NP\}$  be a regular node set and  $A = \{\phi_i | i = 1, \dots, NP\}$  be a set of admissible shape functions which is generated from the smooth window function  $\Phi$ . First we divide node set  $\Lambda$  to three parts and re-indexing as the following for convenience.  $\Lambda_\Gamma = \{x_1, \dots, x_{N_\Gamma}\}$  is the set of boundary node points.  $\Lambda_{\Gamma^*} = \{x_{N_\Gamma+1}, \dots, x_{N_{\Gamma^*}}\}$  is the set of boundary influence node points, i.e., shape functions associated with these node points have non-vanishing values at some boundary node points.  $\Lambda_0 = \{x_{N_{\Gamma^*}+1}, \dots, x_{NP}\}$  is the set of interior node points such that shape function correspond to this node point has compact support in  $\Omega$ . We define the discrete function space  $V^h$  by

$$V^h(\Omega) = \{u(x) = \sum_{i=1}^{NP} c_i \phi_i(x) \mid \phi_i(x) \in A, c_i \in \mathbb{R}, x \in \Omega\}.$$

Also  $V_0^h$  is defined by

$$V_0^h(\Omega) = \{v(x) \in V^h(\Omega) \mid v(x_i) = 0 \text{ for any boundary node } x_i \in \partial\Omega\}.$$

Since it is complicated to describe the function space  $V_0^h$  with shape function set  $A$ , we consider the following linear transformation.

$$\widehat{\phi}_i(x) = \begin{cases} \sum_j d_j^i \phi_j(x) & \text{for } 1 \leq i \leq N_{\Gamma^*} \\ \phi_i(x) & \text{for } N_{\Gamma^*} < i \end{cases} \tag{13}$$

where  $[d_j^i] = [\phi_i(x_j)]^{-1}$ ,  $1 \leq i, j \leq N_{\Gamma^*}$ . We call this transformation as boundary transformation. Note that  $\widehat{\phi}_i(x_k) = \delta_{ik}$ , for  $i, k = 1, \dots, N_{\Gamma^*}$ .

Any function  $u(x) \in V^h(\Omega)$  satisfies

$$\begin{aligned} u(x) &= \sum_{i=1}^{N_{\Gamma^*}} c_i \phi_i(x) + \sum_{i=N_{\Gamma^*}+1}^{NP} c_i \phi_i(x) \\ &= \sum_{i=1}^{N_{\Gamma^*}} \left( \sum_{j=1}^{N_{\Gamma^*}} c_i \phi_i(x_j) \right) \widehat{\phi}_j(x) + \sum_{i=N_{\Gamma^*}+1}^{NP} c_i \phi_i(x) \\ &= \sum_{j=1}^{N_{\Gamma^*}} \left( \sum_{i=1}^{N_{\Gamma^*}} c_i \phi_i(x_j) \right) \widehat{\phi}_j(x) + \sum_{j=N_{\Gamma^*}+1}^{NP} c_j \widehat{\phi}_j(x), \end{aligned}$$

also  $v(x) \in V_0^h(\Omega)$  satisfies

$$v(x) = \sum_{j=N_\Gamma+1}^{N_{\Gamma^*}} \left( \sum_{i=1}^{N_{\Gamma^*}} c_i \phi_i(x_j) \right) \widehat{\phi}_j(x) + \sum_{j=N_{\Gamma^*}+1}^{NP} c_j \widehat{\phi}_j(x),$$

since  $v(x_k) = 0$  for  $k = 1, \dots, N_\Gamma$  and  $\widehat{\phi}_j(x_k) = \delta_{jk}$  for  $1 \leq j, k \leq N_{\Gamma^*}$ . Hence we may consider  $\widetilde{A} = \{\widehat{\phi}_i \mid \phi_i \in A\}$  as a new basis for  $V^h(\Omega)$  and naturally  $\widehat{A} = \{\widehat{\phi}_i \in$

$\tilde{A} | x_i \in \Gamma^* \cup \Gamma_0 \}$  is the basis of  $V_0^h(\Omega)$ , i.e.

$$V_0^h(\Omega) = \{v(x) = \sum_{i=N_\Gamma+1}^{NP} c_i \hat{\phi}_i(x) \mid \hat{\phi}_i \in \hat{A}, c_i \in \mathbb{R}\}.$$

REMARK 2. *There is an issue of invertibility of  $[\phi_i(x_j)]$ . This is closely related to the shape of window function  $\Phi$  and the dilation ratio  $\gamma$ . Basically, the shape function is similar to the window function. Hence we need to design the window function  $\Phi$  and to choose the dilation ratio  $\gamma$  so that the matrix  $[\phi_i(x_j)]$  is invertible.*

DEFINITION 2.4. *Let  $A = \{\phi_i \mid i = 1, \dots, NP\}$  be the admissible set of shape functions generated by the window function  $\Phi \in C_0^m(\mathbb{R}^n)$  for the regular node set  $\Lambda = \{x_i \mid i = 1, \dots, NP\}$ . Here  $\phi_i$ 's satisfy  $m$ -th order consistency, and dilation ratio is  $\gamma$ . We define  $\Phi$  as proper window function of  $m$ -th order and  $\gamma_m$  as proper dilation ratio if*

$$\sum_{j \neq i} |\phi_i(x_j)| < c\rho^{2m},$$

for all  $i = 1, \dots, NP$ .

**Example** Consider  $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ , and the regular node set

$$\Lambda = \left\{ \left( \frac{i}{n}, \frac{j}{n} \right) \mid i = 0, \dots, n \quad j = 0, \dots, n \right\},$$

where  $n$  is some positive constant. Choose the proper window function of second order as

$$\Phi(x, y) = f(x) f(y),$$

where  $f$  is defined by

$$f(x) = \begin{cases} -128x^4 + 1 & 0 \leq x < \frac{1}{4} \\ 128(x - \frac{1}{2})^4 & \frac{1}{4} \leq x < \frac{1}{2} \\ (x - \frac{1}{2})^2(x - 1)^2 & \frac{1}{2} \leq x < 1 \\ 0 & 1 \leq x. \end{cases}$$

Also choose the proper dilation ratio as

$$\gamma = \frac{2}{1-h}, \quad \text{where } h = \frac{1}{n}.$$

As a summary, using boundary transformation, we defined  $V_0^h(\Omega)$  from  $V^h(\Omega)$ . Now we define the test function space for MLSRK method as

$$V_0^h(\Omega) \equiv \{u(x) = \sum_{i=N_\Gamma+1}^{NP} c_i \hat{\phi}_i(x) \mid x \in \Omega, \hat{\phi}_i \in \hat{A}, c_i \in \mathbb{R}\}.$$

The next thing to study is projection for  $C_0(\Omega)$ , the space of continuous function with compact support in  $\Omega$ . For this, we define new projection.

DEFINITION 2.5. *Let  $A = \{\phi_i \mid i = 1, \dots, NP\}$  be the set of admissible shape functions over the regular node set  $\Lambda = \{x_i \mid i = 1, \dots, NP\}$  with window function  $\Phi \in C_0^m(\mathbb{R}^n)$ , here  $\Lambda_\Gamma = \{x_i \mid i = 1, \dots, N_\Gamma\}$  is the set of boundary node. Suppose that  $\hat{A} = \{\hat{\phi}_i \mid i = 1, \dots, NP\}$  is the transformed shape functions using the*

boundary transformation(13). Define a discrete projection of function in  $C_0(\Omega)$  as

$$\widehat{\mathcal{R}}_{\rho,h}^m u(x) \equiv \sum_{i=N_\Gamma+1}^{NP} u(x_i) \widehat{\phi}_i(x). \tag{14}$$

Here  $m, \rho$  and  $h$  represent order of generating polynomial basis, dilation parameter for  $\phi_i$  and node distance.

As a Corollary of Theorem 2.1, we find the following projection error estimate for  $\widehat{\mathcal{R}}_{\rho,h}^m$ .

**COROLLARY 2.1.** Assume that  $\Phi(x) \in C_0^m(\mathbb{R}^n)$  is a proper window function of  $m$ -th order. The domain  $\Omega \subset \mathbb{R}^n$  is bounded open set. Let  $\Lambda = \{x_i | i = 1, \dots, NP\}$  be a regular node set and  $\widehat{A} = \{\widehat{\phi}_i | i = 1, \dots, NP\}$  be the boundary transformed function set of admissible shape function set  $A = \{\phi_i | i = 1, \dots, NP\}$ , here the dilation ratio  $\gamma$  for  $\phi_i$  is proper ratio of  $\Phi(x)$ . Suppose  $v(x) \in C_0^{m+1}(\Omega)$  and

$$m > \frac{n}{p} - 1 .$$

Then the following interpolation estimate holds

$$\|v - \widehat{\mathcal{R}}_{\rho,h}^m v\|_{W^{k,p}(\Omega)} \leq C_k \rho^{m+1-k} \|v\|_{W^{m+1,p}(\Omega)}, \quad \text{for all } 0 \leq k \leq m . \tag{15}$$

*Proof.* Since we have

$$|D_x^\beta v(x) - D_x^\beta \widehat{\mathcal{R}}_{\rho,h}^m v(x)| \leq |D_x^\beta v(x) - D_x^\beta \mathcal{R}_{\rho,h}^m v(x)| + |D_x^\beta \mathcal{R}_{\rho,h}^m v(x) - D_x^\beta \widehat{\mathcal{R}}_{\rho,h}^m v(x)| ,$$

and from Theorem 2.1, it is enough to show

$$\|\mathcal{R}_{\rho,h}^m v - \widehat{\mathcal{R}}_{\rho,h}^m v\|_{W^{k,p}(\Omega)} \leq C_k \rho^{m+1-k} \|v\|_{W^{m+1,p}(\Omega)} .$$

Note that  $v(x) \in C_0^{m+1}(\Omega)$  and  $\phi_i(x) = \widehat{\phi}_i(x)$  for  $x_i \in \Lambda_0$ . Hence we have

$$\begin{aligned} D_x^\beta \mathcal{R}_{\rho,h}^m v(x) - D_x^\beta \widehat{\mathcal{R}}_{\rho,h}^m v(x) &= \sum_{x_i \in \Lambda_{\Gamma^*}} v(x_i) \left( D_x^\beta \phi_i(x) - D_x^\beta \widehat{\phi}_i(x) \right) \\ &= \sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda(x)} v(x_i) \left( D_x^\beta \phi_i(x) - D_x^\beta \widehat{\phi}_i(x) \right) - \sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda^c(x)} v(x_i) \left( \sum_{x_j \in \Lambda(x)} d_j^i D_x^\beta \phi_j(x) \right) , \end{aligned} \tag{16}$$

since  $\phi_i(x) = 0$  for  $x_i \in \Lambda^c(x)$ . Here  $\Lambda^c(x) = \Lambda - \Lambda(x)$ ,  $[d_j^i] = [\phi_i(x_j)]^{-1} \equiv G^{-1}$  and refer previous definitions for notations.

Note that  $\|G^{-1}\|$  is bounded. From the assumption of *proper* window function, smallness of off-diagonal terms for  $G$  is guaranteed. Hence using the consistency of shape functions, we observe

$$G = I - \rho^{2m} G^* , \tag{17}$$

where  $G^*$  is a matrix with  $\|G^*\| = O(1)$ . Now from the Sobolev inequality, we have

$$\begin{aligned} & \left| \sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda(x)} v(x_i) \left( D_x^\beta \phi_i(x) - D_x^\beta \widehat{\phi}_i(x) \right) \right| \\ & \leq \sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda(x)} \sum_{x_j \in \Lambda(x)} \left| v(x_i) (I - G^{-1})_{ij} D_x^\beta \phi_j(x) \right| \\ & \leq c \|v\|_{W^{m+1,p}(\Omega)} \sum_{x_j \in \Lambda(x)} |D_x^\beta \phi_j(x)| \sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda(x)} \left| (I - G^{-1})_{ij} \right| \\ & \leq c \|v\|_{W^{m+1,p}(\Omega)} L \rho^{2m} \sum_{x_j \in \Lambda(x)} |D_x^\beta \phi_j(x)|, \end{aligned}$$

since  $\|G^{-1}\|$  is bounded and

$$\sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda(x)} \left| (G - I)_{ij} \right| \leq L \sum_{i \neq j} |\phi_j(x_i)| < c L \rho^{2m}.$$

For the second part of (16),

$$\begin{aligned} & \left| \sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda^c(x)} v(x_i) \left( \sum_{x_j \in \Lambda(x)} d_j^i D_x^\beta \phi_j(x) \right) \right| \\ & \leq \sum_{x_j \in \Lambda(x)} \sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda^c(x)} |v(x_i) d_j^i D_x^\beta \phi_j(x)| \\ & \leq c \|v\|_{W^{m+1,p}(\Omega)} \sum_{x_j \in \Lambda(x)} \left( |D_x^\beta \phi_j(x)| \sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda^c(x)} |d_j^i| \right) \\ & \leq c \|v\|_{W^{m+1,p}(\Omega)} \rho^{2m} \left( \sum_{x_j \in \Lambda(x)} |D_x^\beta \phi_j(x)| \right). \end{aligned}$$

In above inequalities, the inequality

$$\sum_{x_i \in \Lambda_{\Gamma^*} \cap \Lambda^c(x)} |d_j^i| \leq c \rho^{2m}$$

for  $x_j \in \Lambda(x)$ , is justified from the assumptions and considering Gram-Schmidt process for making  $d_j^i$ .

Hence by following similar argument in Theorem 2.1, we have

$$\|D_x^\beta \mathcal{R}_{\rho,h}^m v - D_x^\beta \widehat{\mathcal{R}}_{\rho,h}^m\|_{W^{k,p}(\Omega)} \leq c \|v\|_{W^{m+1,p}(\Omega)} \rho^{m+1-k} \left( L^2 |\partial\Omega|^{\frac{1}{p}} \rho^{\frac{1}{p}} + L \rho^{\frac{n}{p}} \right),$$

for all  $0 \leq k \leq m$ , and this completes the proof.  $\square$

There is a Sobolev type embedding theorem for  $\widehat{\mathcal{R}}_{\rho,h}^m$  also, and we state it as a corollary of Theorem 2.2.

COROLLARY 2.2. Suppose  $v \in C_0^1(\Omega)$  and the proper window function  $\Phi \in C_0^m(\mathbb{R}^n)$ ,  $m \geq 1$ . If  $p > n$  holds, then we have the following inequalities

$$\|\widehat{\mathcal{R}}_{\rho,h}^m v\|_{L^p} \leq C \|\nabla v\|_{L^p}, \tag{18}$$

$$\sup_{\Omega} |\widehat{\mathcal{R}}_{\rho,h}^m v| \leq C |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|\nabla v\|_{L^p} \tag{19}$$

The above corollary can be proved directly from Theorem 2.2 by considering the difference  $\nabla \mathcal{R}_{\rho,h}^m v(x) - \widehat{\nabla} \mathcal{R}_{\rho,h}^m v(x)$ . Or one can prove it using Corollary 2.1 and the Poincaré inequality.

### 3. Applications of the MLSRK Method to Incompressible Flows.

In this section, we study the stationary incompressible flow with zero velocity on the boundary. Throughout this section, we suppose the following. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary.  $\Lambda^V = \{x_i^V \mid i = 1, \dots, NP\}$  is a *regular* node set in  $\Omega$ , and  $A^V = \{\phi_i \mid i = 1, \dots, NP\}$  is an admissible shape function set with  $m$ -th order consistency. Shape function set  $A^V$  is made by *proper* window function  $\Phi(x) \in C_0^{m+1}(\mathbb{R}^n)$  of  $m$ -th order and *proper* dilation ratio  $\gamma$ .

After re-indexing, let  $\widehat{\Lambda}^V = \{x_i^V \mid i = 1, \dots, \widehat{NP}\}$  be a set of interior node and boundary influence node of  $\Lambda^V$ , and  $\widehat{A}^V = \{\widehat{\phi}_i \mid i = 1, \dots, \widehat{NP}\}$  be a boundary transformed shape function set of  $A^V$ . Hence  $\widehat{A}^V$  is a basis of  $V_0^h(\Omega)$ , where  $V_0^h(\Omega)$  is a test function space as defined in the previous section.  $\widehat{\Lambda}^V$  and  $\widehat{A}^V$  will be used for velocity approximation. Note that even transformed shape function  $\widehat{\phi}_i \in \widehat{A}^V$  does not vanish on the entire boundary  $\partial\Omega$ , it has zero value only on the boundary node points. But the  $L_\infty$  norm of  $\widehat{\phi}_i \in \widehat{A}^V$  is sufficiently small on the boundary of  $\Omega$  so that  $\|\widehat{\phi}_i\|_{L_\infty(\partial\Omega)} \leq c\rho^{2m}$ , as long as the *proper* window function is used for shape functions.

For the pressure, let  $\Lambda^P = \{x_j^P \mid j = 1, \dots, MP\}$  be a *regular* node set in  $\Omega$ , and  $A^P = \{\psi_j \mid j = 1, \dots, MP\}$  be an admissible shape function set with  $m$ -th order consistency from window function  $\Psi(x) \in C_0^m(\mathbb{R}^n)$ . Here we assume that the characteristic distance for pressure node is compatible to the characteristic distance for velocity node, and so is the characteristic support radius of shape function. Hence the discrete solution space for velocity and pressure will be as followings:

$$[V_0^h(\Omega)]^n = \left\{ \mathbf{U}(x) = \sum_{i=1}^{\widehat{NP}} \mathbf{u}_i \widehat{\phi}_i(x) \mid \widehat{\phi}_i \in \widehat{A}^V, \mathbf{u}_i \in \mathbb{R}^n \right\} \tag{20}$$

$$M^h(\Omega) = \left\{ P(x) = \sum_{j=1}^{MP} p_j \psi_j(x) \mid \psi_j \in A^P, p_j \in \mathbb{R}, \int_{\Omega} P(x) dx = 0 \right\}$$

In addition to the assumption on the node distribution, we assume the technical hypothesis. It is about the coercivity of the bilinear form which is induced from the viscous term of governing equations, i.e. we assume that there is a positive constant  $\beta_0$  independent of  $\rho$  such that

$$\frac{1}{\beta_0} \rho^{n-2} \|a\|^2 \geq \sum_{i,j=N_\Gamma+1}^{NP} \left[ \int_{\Omega} \nabla \widehat{\phi}_i \nabla \widehat{\phi}_j dx \right] a_i a_j \geq \beta_0 \rho^{n-2} \|a\|^2 \tag{21}$$

for all  $a \in \mathbb{R}^{NP-N_\Gamma}$ , where  $NP$  is the number of nodes,  $N_\Gamma$  is the number of boundary nodes and  $n$  is the dimension of space.

We denote the discrete projection for velocity as  $\widehat{R}_{\rho,h}^m$  and for pressure as  $S_{\rho,h}^m$ , in detail, we have

$$\begin{aligned}\widehat{R}_{\rho,h}^m \mathbf{u}(x) &= \sum_{i=1}^{\widehat{NP}} \mathbf{u}(x_i^V) \widehat{\phi}_i(x), \quad \widehat{\phi}_i \in \widehat{A}^V, \\ S_{\rho,h}^m p(x) &= \sum_{j=1}^{MP} p(x_j^P) \psi_j(x), \quad \psi_j \in A^P.\end{aligned}\quad (22)$$

For further analysis, we define the following *inf-sup* condition.

DEFINITION 3.1. *We say the pair of shape function sets  $(\widehat{A}^V, A^P)$  satisfies the inf-sup condition if there exists  $\lambda > 0$  independent of  $\rho$  such that*

$$\sup_{\mathbf{U} \in [V_0^h(\Omega)]^n} \frac{\langle \operatorname{div} \mathbf{U}, P \rangle}{\|\nabla \mathbf{U}\|_{L^2}} \geq \lambda \|P\|_{L^2}, \quad (23)$$

for all  $P \in M^h(\Omega)$ .

We find a sufficient condition for  $(\widehat{A}^V, A^P)$  to satisfy the *inf-sup* condition.

THEOREM 3.1. *Suppose  $(\widehat{A}^V, A^P)$  satisfies the following inequality, for any  $j \in \Phi_i$ ,  $i = 1, \dots, MP$ ,*

$$\left| \int_{\Omega} \psi_i \frac{\partial \widehat{\phi}_j}{\partial x_{\alpha}} dx \right| \geq \alpha_0 \rho^{n-1} + \sum_{k \in \Psi_i} \left| \int_{\Omega} \psi_k \frac{\partial \widehat{\phi}_j}{\partial x_{\alpha}} dx \right|, \quad (24)$$

where  $\alpha_0 > 0$  is a constant and  $\Phi_i$  and  $\Psi_i$  are the index sets defined as

$$\begin{aligned}\Phi_i &= \{l \mid x_l^V \in \operatorname{supp} \psi_i\}, \\ \Psi_i &= \{l \mid l \neq i \text{ and } \operatorname{supp} \psi_l \cap \operatorname{supp} \psi_i \neq \emptyset\}.\end{aligned}$$

Then  $(\widehat{A}^V, A^P)$  satisfies the *inf-sup* condition.

*Proof.* Let  $\{p_1, \dots, p_{MP}\}$  be the pressure coefficients so that the discrete pressure is given. Let  $S_0$  be the index set of pressure nodes. Choose  $p_{a_1}$  such that

$$|p_{a_1}| \geq |p_j|, \quad \text{for all } j \in S_0, \quad (25)$$

and let  $S_{a_1} = \{j \mid \operatorname{supp} \psi_j \cap \operatorname{supp} \psi_{a_1} \neq \emptyset\}$ . Choose  $p_{a_2}$ ,  $a_2 \in S_0 - S_{a_1}$  such that

$$|p_{a_2}| \geq |p_j|, \quad \text{for all } j \in S_0 - S_{a_1}, \quad (26)$$

and let  $S_{a_2} = \{j \mid \operatorname{supp} \psi_j \cap \operatorname{supp} \psi_{a_2} \neq \emptyset\}$ . Choose  $p_{a_3}$ ,  $a_3 \in S_0 - (S_{a_1} \cup S_{a_2})$  such that

$$|p_{a_3}| \geq |p_j|, \quad \text{for all } j \in S_0 - (S_{a_1} \cup S_{a_2}), \quad (27)$$

and let  $S_{a_2} = \{j \mid \operatorname{supp} \psi_j \cap \operatorname{supp} \psi_{a_2} \neq \emptyset\}$ . Continuing this process, we have a subset of pressure coefficients such that

$$\{p_{a_1}, p_{a_2}, \dots, p_{a_L}\}. \quad (28)$$

From our assumption of finite intersection property, we have

$$|p_{a_j}|^2 \geq \frac{1}{H} \sum_{k \in S_{a_j}} |p_k|^2, \quad (29)$$

where  $H$  is the maximum number of elements of  $S_{a_j}$  for all  $j = 1, \dots, L$ . The maximum intersection number  $H$  is independent of the dilation parameter  $\rho$  when we refine or coarsen nodes. Thus, we obtain

$$\left[ \sum_{j=1}^L |p_{a_j}|^2 \right]^{\frac{1}{2}} \geq \frac{1}{\sqrt{H}} \left[ \sum_{i=1}^{MP} |p_i|^2 \right]^{\frac{1}{2}}. \quad (30)$$

Now, we choose the value of  $u_j^\alpha$  adequately where  $u_j^\alpha$  is to be the  $\alpha$ -th component of the coefficient of the velocity shape function  $\widehat{\phi}_j$  for  $j = 1, \dots, NP$ . From our choice of  $a_k$ 's, we note that all of the index sets  $\Phi_{a_k}$ 's are mutually disjoint. For each  $k = 1, \dots, L$ , let us choose  $u_j^\alpha$ 's such that, for all  $j \in \Phi_{a_k}$ ,

$$u_j^\alpha = p_{a_k} \operatorname{sign} \left( \int_{\Omega} \psi_{a_k} \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx \right). \quad (31)$$

Otherwise, i.e., if  $j \notin \bigcup_{k=1}^L \Phi_{a_k}$ , then we choose  $u_j^\alpha = 0$ . Hence we have

$$\begin{aligned} & \sum_{i=1}^{MP} \sum_{j=1}^{NP} \sum_{\alpha=1}^n p_i u_j^\alpha \int_{\Omega} \psi_i \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx \\ &= \sum_{k=1}^L \sum_{j=1}^{NP} \sum_{\alpha=1}^n p_{a_k} u_j^\alpha \int_{\Omega} \psi_{a_k} \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx + \sum_{i \notin \{a_1, \dots, a_L\}} \sum_{j=1}^{NP} \sum_{\alpha=1}^n p_i u_j^\alpha \int_{\Omega} \psi_i \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx \\ &= \sum_{k=1}^L \left( \sum_{j \in \Phi_{a_k}} \sum_{\alpha=1}^n p_{a_k} u_j^\alpha \int_{\Omega} \psi_{a_k} \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx + \sum_{i \notin \{a_1, \dots, a_L\}} \sum_{j \in \Phi_{a_k}} \sum_{\alpha=1}^n p_i u_j^\alpha \int_{\Omega} \psi_i \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx \right) \\ &= \sum_{k=1}^L \sum_{j \in \Phi_{a_k}} \sum_{\alpha=1}^n \left( |p_{a_k}|^2 \left| \int_{\Omega} \psi_{a_k} \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx \right| + \sum_{i \notin \{a_1, \dots, a_L\}} p_i p_{a_k} \left| \int_{\Omega} \psi_i \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx \right| \right) \\ &\geq \sum_{k=1}^L \sum_{j \in \Phi_{a_k}} \sum_{\alpha=1}^n |p_{a_k}|^2 \left( \left| \int_{\Omega} \psi_{a_k} \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx \right| - \sum_{i \notin \{a_1, \dots, a_L\}} \left| \int_{\Omega} \psi_i \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx \right| \right) \\ &\geq \frac{n \alpha_0}{H} \sum_{k=1}^{MP} |p_k|^2. \end{aligned}$$

If we normalize the chosen velocity coefficients by  $\left[ \sum_{k=1}^{MP} |p_k|^2 \right]^{\frac{1}{2}}$  and use (31), then we have

$$\sum_{i=1}^{MP} \sum_{j=1}^{NP} \sum_{\alpha=1}^n p_i \frac{u_j^\alpha}{\left[ \sum_{j=1}^{NP} |u_j^\alpha|^2 \right]^{\frac{1}{2}}} \int_{\Omega} \psi_i \frac{\partial \widehat{\phi}_j}{\partial x_\alpha} dx \geq \frac{n \alpha_0}{H} \left[ \sum_{k=1}^{MP} |p_k|^2 \right]^{\frac{1}{2}}.$$

Therefore, it completes the proof.  $\square$

REMARK 3. *Theorem 3.1 implies that if every support of velocity shape function is located at the place where the pressure shape function is steep then the inf-sup condition is satisfied.*

One can suggest to make node distributions to satisfy *inf-sup condition* as the following. First, make pressure node distribution  $x_i^P$ . For each pressure node point  $x_i^P$ , consider adjacent node points which are included in the support of shape functions at  $x_i^P$ . Now consider mid-points of line segments between  $x_i^P$  and adjacent node points. Adding all of those mid points to pressure node points, velocity node distribution is made.

### 3.1. Stokes Problem.

In this subsection, we study the stationary incompressible Stokes flow with vanishing boundary condition. The governing equations are

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{32}$$

where the solution  $(\mathbf{u}, p) \in H_0^{m+1}(\Omega) \times H^m(\Omega)/\mathbb{R}$ , and the function space  $H^m(\Omega)/\mathbb{R}$  is the set of all  $H^m(\Omega)$ -functions with zero mean in  $\Omega$ . Using the MLSRK method, we prove the existence of the numerical solution and its convergence to the exact solution. For simplicity, we assume  $\nu = 1$ .

First we state weak formulation of the Stokes equations. We define a pair  $(\mathbf{U}, P) \in [V_0^h(\Omega)]^n \times M^h(\Omega)$  is the discrete solution to the Stokes equations (32), if  $(\mathbf{U}, P)$  satisfy

$$\begin{aligned} \nu \int_{\Omega} \nabla\mathbf{U} \nabla\mathbf{V} \, dx - \int_{\Omega} P \nabla\cdot\mathbf{V} \, dx &= \int_{\Omega} \mathbf{f} \cdot \mathbf{V} \, dx \\ \int_{\Omega} \nabla\cdot\mathbf{U} Q \, dx &= 0, \end{aligned} \tag{33}$$

for all  $\mathbf{V} \in [V_0^h(\Omega)]^n$  and  $Q \in M^h(\Omega)$ .

THEOREM 3.2. *Suppose that  $\Omega \subset \mathbb{R}^n$  is bounded domain with smooth boundary, and we follow all notations and assumptions in the beginning of this section. Suppose that  $(\widehat{A}^V, A^P)$  satisfies the inf-sup condition. Then there is a unique discrete solution pair  $(\mathbf{U}, P)$  to the discrete Stokes equations (33).*

*Proof.* It is enough to find coefficient vectors of velocity  $\bar{\mathbf{u}}^\alpha = [\bar{u}_1^\alpha, \dots, \bar{u}_{NP}^\alpha]^T$ , for  $\alpha = 1, \dots, n$  and pressure  $\bar{p} = [p_1, \dots, p_{MP}]^T$  for the solution of (33)

$$\mathbf{U}(x) = \begin{pmatrix} U^1(x) \\ \vdots \\ U^n(x) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{\widehat{NP}} \bar{u}_i^1 \widehat{\phi}_i(x) \\ \vdots \\ \sum_{i=1}^{\widehat{NP}} \bar{u}_i^n \widehat{\phi}_i(x) \end{pmatrix} \quad \text{and} \quad P(x) = \sum_{j=1}^{MP} p_j \psi_j(x).$$



Let us define

$$\begin{aligned} M_{ik} &= \int_{\Omega} \nabla \widehat{\phi}_i \nabla \widehat{\phi}_k \, dx, \\ B_{ij}^{\alpha} &= - \int_{\Omega} \frac{\partial \widehat{\phi}_i}{\partial x_{\alpha}} \psi_j \, dx, \\ f_i^{\alpha} &= \int_{\Omega} f^{\alpha} \widehat{\phi}_i \, dx, \end{aligned}$$

where  $1 \leq i, k \leq \widehat{NP}$ ,  $1 \leq j \leq MP$  and  $\alpha (1 \leq \alpha \leq n)$ . Then we obtain the system of the discrete Stokes equations

$$\begin{aligned} \sum_{k=1}^{\widehat{NP}} M_{ik} \bar{u}_k^{\alpha} + \sum_{j=1}^{MP} B_{ij}^{\alpha} p_j &= f_i^{\alpha} \\ \sum_{j=1}^{NP} B_{jl}^{\alpha} \bar{u}_j^{\alpha} &= 0, \end{aligned}$$

for all  $i = 1, \dots, \widehat{NP}$ ,  $l = 1, \dots, MP$  and  $\alpha = 1, \dots, n$ . By defining the assembled matrices  $\mathbf{M}$  and  $B$  such as

$$\mathbf{M} = \text{Diag}(M, \dots, M), \quad \text{and} \quad B = [B^1, \dots, B^n]^T,$$

the discrete Stokes equations (33) is written by matrix form

$$\begin{bmatrix} \mathbf{M} & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{f}} \\ 0 \end{bmatrix}.$$

Note that  $B^T$  is the transpose of  $B$  and  $\bar{\mathbf{u}} = [\bar{\mathbf{u}}^1, \dots, \bar{\mathbf{u}}^n]^T$ . From the hypothesis (21), we have

$$a^T \mathbf{M}^{-1} a \geq \beta_0 \rho^{2-n} \|a\|^2 \quad (34)$$

for a positive constant  $\beta_0$ , where  $a = [a^1, \dots, a^n]$  with each  $a^{\alpha} \in \mathbb{R}^{\widehat{NP}}$ . Let  $b = (b_1, \dots, b_{MP}) \in \mathbb{R}^{MP}$  be an arbitrary vector such that

$$Q = \sum_{j=1}^{MP} b_j \psi_j(x) \in M^h. \quad (35)$$

Then, the inf-sup condition, the hypothesis (21) and the admissibility of shape functions for pressure imply

$$\begin{aligned} \|Bb\| &= \sup_a \frac{a^T Bb}{\|a\|} \geq \beta_0^{\frac{1}{2}} \rho^{\frac{n}{2}-1} \sup_a \frac{a^T Bb}{(a^T \mathbf{M} a)^{\frac{1}{2}}} \\ &\geq \beta_0^{\frac{1}{2}} \rho^{\frac{n}{2}-1} \lambda \|Q\| \geq \beta_1 \lambda \rho^{n-1} \|b\|, \end{aligned} \quad (36)$$

for some  $\lambda > 0$  and  $\beta_1 > 0$  independent of  $\rho$ , hence, from (34) and (36), there exists some  $\beta_2 > 0$  independent of  $\rho$  such that

$$b^T B^T \mathbf{M}^{-1} B b \geq \beta_2 \rho^n \|b\|^2$$

for any vector  $b$  satisfying (35). Therefore  $B^T \mathbf{M}^{-1} B$  is invertible. Now if we let

$$\bar{p} = (B^T \mathbf{M}^{-1} B)^{-1} B^T \mathbf{M}^{-1} \bar{\mathbf{f}}$$

and

$$\bar{\mathbf{u}} = \mathbf{M}^{-1} (\bar{\mathbf{f}} - B \bar{p}),$$

then the pair  $(\bar{\mathbf{u}}, \bar{p})$  satisfies our discrete Stokes equations (33).

For the uniqueness, we consider two discrete solutions  $(\bar{\mathbf{u}}_1, \bar{p}_1)$  and  $(\bar{\mathbf{u}}_2, \bar{p}_2)$ . Then the residual pair  $(\bar{\mathbf{v}}, \bar{q}) = (\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2, \bar{p}_1 - \bar{p}_2)$  satisfies

$$\begin{aligned}\mathbf{M}\bar{\mathbf{v}} + B\bar{q} &= 0 \\ B^T\bar{\mathbf{v}} &= 0.\end{aligned}$$

Multiplying  $\bar{q}$  on the second equation and  $\bar{\mathbf{v}}$  on the first equation, we have

$$\bar{\mathbf{v}}^T\mathbf{M}\bar{\mathbf{v}} + \bar{\mathbf{v}}^TB\bar{q} = 0, \quad \bar{q}^TB^T\bar{\mathbf{v}} = 0.$$

Since  $\bar{\mathbf{v}}^TB\bar{q}$  and  $\bar{q}^TB^T\bar{\mathbf{v}}$  are scalars, we conclude that

$$\bar{\mathbf{v}}^TB\bar{q} = \bar{q}^TB^T\bar{\mathbf{v}} = 0.$$

Thus we have

$$\bar{\mathbf{v}}^T\mathbf{M}\bar{\mathbf{v}} = 0.$$

Since  $\mathbf{M}$  is positive definite, we have  $\bar{\mathbf{v}} = 0$ . From the inequality (36),

$$\beta_1\lambda\rho^{n-1}\|\bar{q}\| \leq \|B\bar{q}\| = \|\mathbf{M}\bar{\mathbf{v}}\| = 0,$$

which implies  $\bar{q} = 0$ . Therefore the uniqueness holds.  $\square$

Let  $(\mathbf{u}, p)$  be the solution of the Stokes equations (32), and  $(\mathbf{U}, P)$  be a discrete solution of (33). Then we have error equations

$$\begin{aligned}\int_{\Omega} (\nabla\mathbf{U} - \nabla\mathbf{u}) \cdot \nabla\mathbf{V} \, dx + \int_{\partial\Omega} \frac{\partial\mathbf{u}}{\partial\mathbf{n}} \mathbf{V} \, d\Gamma \\ - \int_{\Omega} (P - p) \nabla \cdot \mathbf{V} \, dx - \int_{\partial\Omega} p \mathbf{V} \cdot \mathbf{n} \, d\Gamma = 0, \quad (37)\end{aligned}$$

$$\int_{\Omega} \nabla \cdot \mathbf{U} Q \, dx = 0, \quad (38)$$

where  $\mathbf{V} \in [V_0^h(\Omega)]^n$ ,  $Q \in M^h(\Omega)$ .

Choosing the test function as  $\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}$ , we have

$$\begin{aligned}\int_{\Omega} \nabla (\mathbf{U} - \mathbf{u}) \cdot \nabla (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, dx + \int_{\partial\Omega} \frac{\partial\mathbf{u}}{\partial\mathbf{n}} (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, d\Gamma \\ - \int_{\Omega} (P - p) \nabla \cdot (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, dx - \int_{\partial\Omega} p (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \cdot \mathbf{n} \, d\Gamma = 0.\end{aligned}$$

Adding and subtracting  $\nabla\mathbf{u}$  to  $\nabla(\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})$ , we have

$$\begin{aligned}\int_{\Omega} |\nabla\mathbf{U} - \nabla\mathbf{u}|^2 \, dx &= - \int_{\Omega} \nabla (\mathbf{U} - \mathbf{u}) \cdot \nabla (\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, dx \\ &+ \int_{\Omega} (P - p) \nabla \cdot \mathbf{U} \, dx - \int_{\Omega} (P - p) \nabla \cdot \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u} \, dx \\ &- \int_{\partial\Omega} \frac{\partial\mathbf{u}}{\partial\mathbf{n}} (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, d\Gamma + \int_{\partial\Omega} p (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \cdot \mathbf{n} \, d\Gamma \\ &\equiv I + II + III + IV + V.\end{aligned}$$

From the Hölder inequality, we have

$$|I| \leq \|\nabla (\mathbf{U} - \mathbf{u})\|_{L^2} \|\nabla (\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2}. \quad (39)$$

Using the divergence free condition (38) of  $\mathbf{U}$  in the discrete sense and  $\nabla \cdot \mathbf{u} = 0$  in the continuous sense, we obtain

$$\begin{aligned} II &= \int_{\Omega} [(P - \mathcal{S}_{\rho,h}^m p) + (\mathcal{S}_{\rho,h}^m p - p)] \nabla \cdot \mathbf{U} \, dx \\ &= \int_{\Omega} (\mathcal{S}_{\rho,h}^m p - p) \nabla \cdot \mathbf{U} \, dx \\ &= \int_{\Omega} (\mathcal{S}_{\rho,h}^m p - p) \nabla \cdot (\mathbf{U} - \mathbf{u}) \, dx. \end{aligned}$$

Again using the Hölder inequality,

$$|II| \leq \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} \|\mathcal{S}_{\rho,h}^m p - p\|_{L^2}. \quad (40)$$

Now using  $\nabla \cdot \mathbf{u} = 0$  in (32), we have

$$|III| \leq (\|P - \mathcal{S}_{\rho,h}^m p\|_{L^2} + \|p - \mathcal{S}_{\rho,h}^m p\|_{L^2}) \|\nabla(\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2}. \quad (41)$$

From the error equation (37) and the inf-sup condition, we can estimate the pressure term in (41) as

$$\begin{aligned} & \frac{\int_{\Omega} \nabla(\mathbf{U} - \mathbf{u}) \nabla \zeta \, dx + \int_{\Omega} (p - \mathcal{S}_{\rho,h}^m p) \nabla \cdot \zeta \, dx - \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \zeta \, d\Gamma + \int_{\partial\Omega} p \zeta \cdot \mathbf{n} \, d\Gamma}{\|\nabla \zeta\|_{L^2}} \\ &= \frac{\int_{\Omega} (P - \mathcal{S}_{\rho,h}^m p) \nabla \cdot \zeta \, dx}{\|\nabla \zeta\|_{L^2}} \geq \lambda \|P - \mathcal{S}_{\rho,h}^m p\|_{L^2} \end{aligned}$$

for some  $\zeta = \sum_{i=1}^{\widehat{N}P} \mathbf{c}_i \widehat{\phi}_i$ . Now note that

$$\begin{aligned} \left| \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \zeta \, d\Gamma \right| &\leq \sum_i |\mathbf{c}_i| \left| \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \widehat{\phi}_i \, d\Gamma \right| \\ &\leq C \|\mathbf{c}\|_{l^2} \rho^{2m} \|\mathbf{u}\|_{H^2} \\ &\leq C \rho^{2m - \frac{n}{2} + 1} \|\nabla \zeta\|_{L^2} \|\mathbf{u}\|_{H^2}, \end{aligned} \quad (42)$$

from the assumption of *proper* window function and the coercivity assumption (21). Likewise, we have

$$\left| \int_{\partial\Omega} p \zeta \cdot \mathbf{n} \, d\Gamma \right| \leq C \rho^{2m - \frac{n}{2} + 1} \|\nabla \zeta\|_{L^2} \|p\|_{H^1}, \quad (43)$$

Therefore we have the following pressure estimate

$$\|P - \mathcal{S}_{\rho,h}^m p\|_{L^2} \leq \frac{1}{\lambda} \left\{ \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} + \|p - \mathcal{S}_{\rho,h}^m p\|_{L^2} + C \rho^{2m - \frac{n}{2} + 1} (\|\mathbf{u}\|_{H^2} + \|p\|_{H^1}) \right\}. \quad (44)$$

From the above inequality (44) with (41), we obtain

$$\begin{aligned} |III| &\leq C \left\{ \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} + \|p - \mathcal{S}_{\rho,h}^m p\|_{L^2} \right. \\ &\quad \left. + \rho^{2m - \frac{n}{2} + 1} (\|\mathbf{u}\|_{H^2} + \|p\|_{H^1}) \right\} \|\nabla(\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2}. \end{aligned} \quad (45)$$

Now for the estimate of  $IV$ , suppose  $(\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) = \sum_i \alpha_i \widehat{\phi}_i$ ,  $\alpha = (\alpha_1, \dots, \alpha_{NP})$ . Then we have,

$$\begin{aligned}
|IV| &= \left| \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, d\Gamma \right| \\
&\leq \sum_i |\alpha_i| \left| \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \widehat{\phi}_i \, d\Gamma \right| \\
&\leq \|\alpha\|_{l^2} \rho^{2m} \|\mathbf{u}\|_{H^2} \\
&\leq C \rho^{2m - \frac{n}{2} + 1} \|\nabla (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2} \|\mathbf{u}\|_{H^2} \\
&\leq C \rho^{2m - \frac{n}{2} + 1} \left\{ \|\nabla (\mathbf{U} - \mathbf{u})\|_{L^2} + \|\nabla (\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2} \right\} \|\mathbf{u}\|_{H^2}, \quad (46)
\end{aligned}$$

from the assumption of *proper* window function, the coercivity assumption (21). Likewise, we have

$$|V| \leq C \rho^{2m - \frac{n}{2} + 1} \left\{ \|\nabla (\mathbf{U} - \mathbf{u})\|_{L^2} + \|\nabla (\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2} \right\} \|p\|_{H^1}. \quad (47)$$

Applying interpolation theorem (9) to the pressure term  $\|p - \mathcal{S}_{\rho,h}^m p\|_{L^2}$  and applying (15) to  $\|\nabla (\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2}$ , and combining all the above estimates (39), (40), (45), (46) and (47) we obtain the following  $H^1$  estimate:

$$\|\nabla (\mathbf{U} - \mathbf{u})\|_{L^2} \leq C \rho^m (\|\mathbf{u}\|_{H^{m+1}} + \|p\|_{H^m}).$$

Now from the coercivity (8), (21) and interpolation theorem (15),

$$\|\mathbf{U} - \mathbf{u}\|_{L^2} \leq C \rho \|\nabla (\mathbf{U} - \mathbf{u})\|_{L^2} + C \rho^{m+1} \|\mathbf{u}\|_{H^{m+1}}.$$

Also from the pressure estimate (44) and the interpolation inequality for the pressure we get

$$\|P - p\|_{L^2} \leq C \rho^m (\|\mathbf{u}\|_{H^{m+1}} + \|p\|_{H^m}).$$

Therefore we proved the following Theorem.

**THEOREM 3.3.** *Suppose that  $(\mathbf{u}, p) \in H_0^{m+1}(\Omega) \times H^m(\Omega)/\mathbb{R}$  is the solution of the Stokes problem (32) and  $(\mathbf{U}, P) \in (C_0^{m+1}(\Omega) \times C^m(\Omega)) \cap ([V_0^h(\Omega)]^n \times M^h(\Omega))$  is the solution of the discrete Stokes problem (33). If  $m > \frac{n}{2} - 1$ , then the following error estimate holds*

$$\begin{aligned}
\|\mathbf{U} - \mathbf{u}\|_{L^2(\Omega)} + \rho \|\nabla (\mathbf{U} - \mathbf{u})\|_{L^2(\Omega)} + \rho \|P - p\|_{L^2(\Omega)} \\
\leq C \rho^{m+1} (\|\mathbf{u}\|_{H^{m+1}(\Omega)} + \|p\|_{H^m(\Omega)}). \quad (48)
\end{aligned}$$

**REMARK 4.** *When we apply the meshfree method which use the Galerkin formulation to the Dirichlet problem, terms involving boundary integral are not exactly zero. But as we have seen in the proof of the above theorem, those terms involving boundary integral can be controlled by choosing the window function and dilation parameter. Indeed, estimates for terms involving boundary integrals are somehow trivial, but we include for the completeness.*

### 3.2. Navier-Stokes Problem.

In this subsection we study the stationary incompressible Navier-Stokes equations with zero boundary condition for the space dimension  $n = 2$  and  $3$ . Let  $(\mathbf{u}, p) \in H_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$  be the solution of the Navier-Stokes equations

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (49)$$

The necessary regularity of  $\mathbf{f}$  will be assumed. We will assume the same hypotheses for the velocity and the pressure nodes as in previous section, also we follow all notations and assumptions.

The discrete solution  $(\mathbf{U}, P) \in [V_0^h(\Omega)]^n \times M^h$  satisfies

$$\begin{aligned} \nu \int_{\Omega} \nabla \mathbf{U} \nabla \mathbf{V} \, dx + \int_{\Omega} (\mathbf{U} \cdot \nabla \mathbf{U}) \mathbf{V} \, dx - \int_{\Omega} P \cdot \nabla \cdot \mathbf{V} \, dx &= \int_{\Omega} \mathbf{f} \mathbf{V} \, dx \\ \int_{\Omega} \nabla \cdot \mathbf{U} Q \, dx &= 0, \end{aligned} \quad (50)$$

for all  $\mathbf{V} \in [V_0^h(\Omega)]^n$ ,  $Q \in M^h$ .

For the existence of discrete solution we prepare the following. Indeed, with the admissibility condition of shape functions and Sobolev type inequality (2.2), we have an  $L^2$  inverse type inequality which provides a fixed point theorem.

LEMMA 3.1. *Assume the space dimension  $n = 2$  or  $3$ . Suppose  $F(x) \in V_0^h(\Omega)$ , then*

$$\|F\|_{L^\infty} \leq \frac{c(\varepsilon)}{\rho^{\frac{1+\varepsilon}{2}}} \|\nabla F\|_{L^2}$$

for given  $\varepsilon > 0$  and for some constant  $c(\varepsilon)$  depending on  $\varepsilon$ .

*Proof.* Note that  $F(x)$  is represented by

$$F(x) = \sum_{i=1}^{NP} f_i \phi_i, \quad \text{for all } \phi_i \in A^V$$

For fixed  $\varepsilon$ , from the Sobolev type inequality (19) and finite overlapping property of shape function, we have

$$\begin{aligned} \|F\|_{L^\infty} &\leq C(\varepsilon) \left[ \int_{\Omega} |\nabla F|^{3+\varepsilon} \, dx \right]^{\frac{1}{3+\varepsilon}} = C(\varepsilon) \left[ \int_{\Omega} \left| \sum_{x_i \in \Lambda(x)} f_i \nabla \phi_i(x) \right|^{3+\varepsilon} \, dx \right]^{\frac{1}{3+\varepsilon}} \\ &\leq C(\varepsilon) L \left[ \sum_{i=1}^{NP} |f_i|^{3+\varepsilon} \int_{\Omega} |\nabla \phi_i(x)|^{3+\varepsilon} \, dx \right]^{\frac{1}{3+\varepsilon}} \leq C(\varepsilon) L \left[ \sum_{i=1}^{NP} |f_i|^{3+\varepsilon} \frac{\rho^n}{\rho^{3+\varepsilon}} \right]^{\frac{1}{3+\varepsilon}} \\ &\leq C(\varepsilon) L \left[ \frac{\|F\|_{L^\infty}}{\rho} \right]^{\frac{1+\varepsilon}{3+\varepsilon}} \left[ \sum_i |f_i|^2 \frac{\rho^n}{\rho^2} \right]^{\frac{1}{3+\varepsilon}} \leq C(\varepsilon) L \left[ \frac{\|F\|_{L^\infty}}{\rho} \right]^{\frac{1+\varepsilon}{3+\varepsilon}} \|\nabla F\|_{L^2}^{\frac{2}{3+\varepsilon}}, \end{aligned}$$

where  $\Lambda(x) = \{x_i \in \Lambda \mid x \in \text{supp}(\phi_i(x)) \cap \bar{\Omega}\}$ . Hence this implies

$$\|F\|_{L^\infty} \leq C(\varepsilon) \rho^{-\frac{1+\varepsilon}{2}} \|\nabla F\|_{L^2}.$$

□

THEOREM 3.4. *For the space dimension  $n = 2$  or  $3$ , let  $\mathbf{w} \in H_0^1(\Omega)$  be divergence free in the discrete sense, i.e.,*

$$\int_{\Omega} \nabla \cdot \mathbf{w} \psi_j dx = 0, \quad j = 1, \dots, MP$$

and  $\mathbf{F} \in V_0^h(\Omega)$  be a given function. Then we have

$$\left| \int_{\Omega} |\mathbf{F}|^2 \nabla \cdot \mathbf{w} dx \right| \leq C \rho^{\frac{1-\varepsilon}{2}} \|\nabla \mathbf{w}\|_{L^2} \|\nabla \mathbf{F}\|_{L^2}^2 \quad (51)$$

for any  $\varepsilon \in (0, 1)$ , where  $\rho > 0$  is the dilation parameter of the shape function.

*Proof.* We let  $F(x) = \sum_{i=1}^{\widehat{NP}} f_i \widehat{\phi}_i(x)$ . Assume  $\mathcal{S}_{\rho,h}^m$  is the discrete projection in terms of the shape functions  $\psi_j$ 's. Since  $\nabla \cdot \mathbf{w} = 0$  in the discrete sense,

$$\int_{\Omega} \nabla \cdot \mathbf{w} \mathcal{S}_{\rho,h}^m(|\mathbf{F}|^2) dx = 0$$

and thus we can write

$$\int_{\Omega} \nabla \cdot \mathbf{w} |\mathbf{F}|^2 dx = \int_{\Omega} \nabla \cdot \mathbf{w} (|\mathbf{F}|^2 - \mathcal{S}_{\rho,h}^m(|\mathbf{F}|^2)) dx.$$

Using Hölder inequality and interpolation inequality (9), we obtain

$$\begin{aligned} \left| \int_{\Omega} \nabla \cdot \mathbf{w} [|\mathbf{F}|^2 - \mathcal{S}_{\rho,h}^m(|\mathbf{F}|^2)] dx \right| &\leq \|\nabla \cdot \mathbf{w}\|_{L^2} \| |\mathbf{F}|^2 - \mathcal{S}_{\rho,h}^m(|\mathbf{F}|^2) \|_{L^2} \\ &\leq C \rho \|\nabla \cdot \mathbf{w}\|_{L^2} \|\nabla(|\mathbf{F}|^2)\|_{L^2} \\ &\leq C \rho \|\nabla \cdot \mathbf{w}\|_{L^2} \|\mathbf{F}\|_{L^\infty} \|\nabla \mathbf{F}\|_{L^2}. \end{aligned}$$

Now from the previous lemma, we have

$$\|\mathbf{F}\|_{L^\infty} \leq \frac{c(\varepsilon)}{\rho^{\frac{1+\varepsilon}{2}}} \|\nabla \mathbf{F}\|_{L^2}$$

and hence we have

$$\left| \int_{\Omega} |\mathbf{F}|^2 \nabla \cdot \mathbf{w} dx \right| \leq C \rho^{\frac{1-\varepsilon}{2}} \|\nabla \mathbf{w}\|_{L^2} \|\nabla \mathbf{F}\|_{L^2}^2$$

□

THEOREM 3.5. *There exists  $\rho_0 > 0$  depending on  $\|\mathbf{f}\|_{H^{-1}}$  and  $\nu$  such that if  $0 < \rho < \rho_0$ , then there is a discrete solution  $(\mathbf{U}, P)$  to the discrete Navier-Stokes equations (50) for space dimension  $n = 2, 3$ .*

*Proof.* We only prove the three dimensional case. Indeed the two dimensional case is easier and we omit the proof. Let us introduce a solution map whose fixed point is a solution to the discrete Navier-Stokes equations. We define the finite dimensional function space

$$\mathcal{W} = \left\{ \mathbf{w} = \sum_{i=1}^{\widehat{NP}} \mathbf{c}_i \widehat{\phi}_i \mid \mathbf{c}_i \in \mathbb{R}^3, \widehat{\phi}_i \in \widehat{AV}, \int_{\Omega} \psi_i \nabla \cdot \mathbf{w} dx = 0 \quad \forall \psi_i \in A^P \right\}.$$

Let  $\mathbf{W} \in \mathcal{W}$  be fixed. We define a map  $\mathcal{L} : \mathcal{W} \rightarrow \mathcal{W}$  such that  $\mathbf{U} = \mathcal{L}(\mathbf{W})$  satisfying

$$\begin{aligned} \nu \int_{\Omega} \nabla \mathbf{U} \nabla \mathbf{V} \, dx + \int_{\Omega} (\mathbf{W} \cdot \nabla \mathbf{U}) \mathbf{V} \, dx - \int_{\Omega} P \nabla \cdot \mathbf{V} \, dx &= \int_{\Omega} \mathbf{f} \mathbf{V} \, dx \\ \int_{\Omega} Q \nabla \cdot \mathbf{U} \, dx &= 0 \end{aligned}$$

for all  $\mathbf{V} \in [V_0^h(\Omega)]^3$ ,  $Q \in M^h(\Omega)$ . Define a subset  $\mathcal{W}(\gamma) \subset \mathcal{W}$  for each  $\gamma > 0$  by

$$\mathcal{W}(\gamma) = \{\mathbf{w} \in \mathcal{W} \mid \|\nabla \mathbf{w}\|_{L^2} \leq \gamma\}.$$

First we claim the following. For given  $\nu > 0$  and  $\gamma > 0$ , there is  $\rho_0 > 0$  so that if  $\rho < \rho_0$  then for each  $\mathbf{w} \in \mathcal{W}(\gamma)$  the matrix  $M(\mathbf{w})$  is invertible. Here the matrix  $M(\mathbf{w})$ , that is associated with the map  $\mathcal{L}$ , is defined by

$$M_{ij}(\mathbf{w}) = \nu \int_{\Omega} \nabla \hat{\phi}_i \nabla \hat{\phi}_j \, dx + \int_{\Omega} (\mathbf{w} \cdot \nabla \hat{\phi}_i) \hat{\phi}_j \, dx.$$

Indeed, for any given  $\mathbf{U} = [U^1, \dots, U^n]$  where  $U^\alpha = \sum_{i=1}^{\widehat{NP}} c_i^\alpha \hat{\phi}_i$  and  $\alpha = 1, 2, 3$ , we get

$$\begin{aligned} \mathbf{c}^{\alpha T} M(\mathbf{w}) \mathbf{c}^\alpha &= \sum_{i=1}^{\widehat{NP}} \sum_{j=1}^{\widehat{NP}} c_i^\alpha c_j^\alpha M_{ij} \\ &= \nu \int_{\Omega} |\nabla \mathbf{U}^\alpha|^2 \, dx - \frac{1}{2} \int_{\Omega} |\mathbf{U}^\alpha|^2 \nabla \cdot \mathbf{w} \, dx. \end{aligned}$$

Since  $\nabla \cdot \mathbf{w} = 0$  in the discrete sense, from the estimate (51), the following is obtained

$$\left| \int_{\Omega} \nabla \cdot \mathbf{w} |\mathbf{U}^\alpha|^2 \, dx \right| \leq C \rho^{\frac{1-\varepsilon}{2}} \|\nabla \mathbf{w}\|_{L^2} \|\nabla \mathbf{U}^\alpha\|_{L^2}^2. \quad (52)$$

Hence, if  $\rho$  is chosen so small that

$$C \rho^{\frac{1-\varepsilon}{2}} \|\nabla \mathbf{w}\|_{L^2} \leq C \rho^{\frac{1-\varepsilon}{2}} \gamma \leq \nu, \quad (53)$$

then we have

$$\mathbf{c}^{\alpha T} M(\mathbf{w}) \mathbf{c}^\alpha \geq \frac{\nu}{2} \int_{\Omega} |\nabla \mathbf{U}^\alpha|^2 \, dx \geq C \nu \rho^{n-2} |\mathbf{c}^\alpha|^2.$$

Since  $\mathbf{c}^\alpha$  is chosen arbitrarily,  $M(\mathbf{w})$  is invertible. If we define the assembled matrices  $\mathbf{M}(\mathbf{w})$  and  $B$  as in the case of the Stokes equations (see the proof of Theorem 3.2 for the definition of  $B$ ), we obtain

$$\begin{bmatrix} \mathbf{M}(\mathbf{w}) & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{f}} \\ 0 \end{bmatrix}.$$

Since  $\mathbf{M}(\mathbf{w})$  is invertible, the above matrix equation has a unique solution as long as  $B$  satisfies the inf-sup condition (23).

Now we have to prove  $\mathbf{U} \in \mathcal{W}(\gamma)$  for sufficiently small  $\gamma > 0$ . From the energy estimates, if  $C \rho^{\frac{1-\varepsilon}{2}} \gamma \leq \frac{\nu}{2}$ , we obtain

$$\int_{\Omega} |\nabla \mathbf{U}|^2 \, dx \leq \frac{2}{\nu} \|\mathbf{f}\|_{H^{-1}} \|\nabla \mathbf{U}\|_{L^2}. \quad (54)$$

Thus the following is derived

$$\|\nabla \mathbf{U}\|_{L^2} \leq \frac{2}{\nu} \|\mathbf{f}\|_{H^{-1}}.$$

The inequality (54) is derived from the estimates (52) and the following identities

$$\begin{aligned} \nu \int_{\Omega} |\nabla \mathbf{U}|^2 dx &= \int_{\Omega} \mathbf{f} \cdot \mathbf{U} dx - \int_{\Omega} (\mathbf{w} \cdot \nabla \mathbf{U}) \cdot \mathbf{U} dx \\ &= \int_{\Omega} \mathbf{f} \cdot \mathbf{U} dx + \frac{1}{2} \int_{\Omega} \nabla \cdot \mathbf{w} |\mathbf{U}|^2 dx. \end{aligned}$$

Therefore, if we choose  $\gamma = \frac{2}{\nu} \|\mathbf{f}\|_{H^{-1}}$  and  $\rho_0 \leq \left[ \frac{\nu^2}{4C \|\mathbf{f}\|_{H^{-1}}} \right]^{\frac{2}{1-\varepsilon}}$ , then the map  $\mathcal{L} : \mathcal{W}(\gamma) \rightarrow \mathcal{W}(\gamma)$  is an into map. To complete our proof, we have to show that the map  $\mathcal{L}$  is continuous. Let  $\mathbf{W}_1$  and  $\mathbf{W}_2$  be arbitrary two functions in  $\mathcal{W}(\gamma)$  and  $\mathbf{U}_1 = \mathcal{L}(\mathbf{W}_1)$  and  $\mathbf{U}_2 = \mathcal{L}(\mathbf{W}_2)$ . If we let  $\bar{\mathbf{W}} = \mathbf{W}_2 - \mathbf{W}_1$  and  $\bar{\mathbf{U}} = \mathbf{U}_2 - \mathbf{U}_1$ , then we have

$$-\nu \Delta \bar{\mathbf{U}} + \bar{\mathbf{W}} \cdot \nabla \mathbf{U}_2 + \mathbf{W}_1 \cdot \nabla \bar{\mathbf{U}} + \nabla(P_2 - P_1) = 0.$$

Taking  $\bar{\mathbf{U}}$  as a test function and using (52), the following estimate is obtained

$$\begin{aligned} \nu \|\nabla \bar{\mathbf{U}}\|_{L^2}^2 &= - \int_{\Omega} (\bar{\mathbf{W}} \cdot \nabla \mathbf{U}_2) \cdot \bar{\mathbf{U}} dx + \frac{1}{2} \int_{\Omega} \nabla \cdot \mathbf{W}_1 |\bar{\mathbf{U}}|^2 dx \\ &\leq \|\bar{\mathbf{W}}\|_{L^3} \|\bar{\mathbf{U}}\|_{L^6} \|\nabla \mathbf{U}_2\|_{L^2} + \frac{1}{2} C \rho^{\frac{1-\varepsilon}{2}} \gamma \|\nabla \bar{\mathbf{U}}\|_{L^2}^2. \end{aligned}$$

If  $0 < \rho < \rho_0$ , from the Sobolev embedding theorem, we obtain

$$\begin{aligned} \frac{\nu}{2} \|\nabla \bar{\mathbf{U}}\|_{L^2}^2 &\leq C |\Omega|^{\frac{1}{6}} \|\bar{\mathbf{W}}\|_{L^6} \|\nabla \mathbf{U}_2\|_{L^2} \|\bar{\mathbf{U}}\|_{L^6} \\ &\leq C |\Omega|^{\frac{1}{6}} \gamma \|\nabla \bar{\mathbf{W}}\|_{L^2} \|\nabla \bar{\mathbf{U}}\|_{L^2}. \end{aligned}$$

Consequently, we have shown the continuity of the map  $\mathcal{L}$  by

$$\|\nabla(\mathcal{L}(\mathbf{W}_2) - \mathcal{L}(\mathbf{W}_1))\|_{L^2} \leq C |\Omega|^{\frac{1}{6}} \frac{\gamma}{\nu} \|\nabla(\mathbf{W}_2 - \mathbf{W}_1)\|_{L^2}.$$

Now by the Leray-Schauder fixed point theorem, we prove our existence theorem for the incompressible Navier-Stokes equations.  $\square$

REMARK 5. *Although we could show the existence of discrete solutions to the discrete Navier-Stokes equations without any restrictions on the size of the external force  $\mathbf{f}$  as long as  $\rho$  is small, to prove uniqueness, we need an intrinsic smallness condition for the size of  $H^{-1}$  norm of the external force  $\mathbf{f}$  for the three dimension.*

Now let us consider the uniqueness of solution for the discrete Navier-Stokes equations in the case  $n = 3$ . Suppose  $\mathbf{U}_1$  and  $\mathbf{U}_2$  be two solutions of the discrete Navier-Stokes equations. If we assume that  $\frac{2\|\mathbf{f}\|_{H^{-1}}}{\nu^2}$  is sufficiently small, then, from the energy estimate, we have

$$\nu \|\nabla(\mathbf{U}_1 - \mathbf{U}_2)\|_{L^2}^2 \leq C \left(1 + \rho^{\frac{1-\varepsilon}{2}}\right) \frac{2\|\mathbf{f}\|_{H^{-1}}}{\nu} \|\nabla(\mathbf{U}_1 - \mathbf{U}_2)\|_{L^2}^2.$$

The above inequality implies that the solution is unique if

$$C(1 + \rho^{\frac{1-\varepsilon}{2}}) \frac{2\|\mathbf{f}\|_{H^{-1}}}{\nu^2} \leq 1.$$

Let  $(\mathbf{U}, P)$  is a solution of the discrete Navier-Stokes equations (50), and  $(\mathbf{u}, p)$  is a solution of the Navier-Stokes equations (49). Then we have the following error



equations

$$\begin{aligned}
& \nu \int_{\Omega} \nabla(\mathbf{U} - \mathbf{u}) \nabla \mathbf{V} \, dx + \nu \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \mathbf{V} \, d\Gamma + \int_{\Omega} (\mathbf{U} - \mathbf{u}) \cdot \nabla \mathbf{U} \mathbf{V} \, dx \\
& + \int_{\Omega} \mathbf{u} \cdot \nabla(\mathbf{U} - \mathbf{u}) \mathbf{V} \, dx - \int_{\Omega} (P - p) \nabla \cdot \mathbf{V} - \int_{\partial\Omega} p \mathbf{V} \cdot \mathbf{n} \, d\Gamma = 0 \quad (55) \\
& \int_{\Omega} \nabla \cdot (\mathbf{U} - \mathbf{u}) Q \, dx = 0,
\end{aligned}$$

for all  $\mathbf{V} \in [V_0^h(\Omega)]^n$  and  $Q \in M^h(\Omega)$ . Taking  $\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}$  as a test function to the discrete error equation (55), we have the equality

$$\begin{aligned}
& \nu \int_{\Omega} |\nabla(\mathbf{U} - \mathbf{u})|^2 \, dx = -\nu \int_{\Omega} \nabla(\mathbf{U} - \mathbf{u}) \nabla(\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, dx \\
& - \int_{\Omega} (\mathbf{U} - \mathbf{u}) \cdot \nabla \mathbf{U} \cdot (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, dx - \int_{\Omega} \mathbf{u} \cdot \nabla(\mathbf{U} - \mathbf{u}) \cdot (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, dx \\
& + \int_{\Omega} (P - p) \nabla \cdot (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, dx - \nu \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, d\Gamma \\
& + \int_{\partial\Omega} p (\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \cdot \mathbf{n} \, d\Gamma \\
& \equiv I + II + III + IV + V + VI.
\end{aligned}$$

The first term  $I$  is estimated as follows

$$|I| \leq \nu \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} \|\nabla(\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2}.$$

Also we can obtain the following estimates for  $II$ ,  $III$  and  $V$ ,

$$\begin{aligned}
|II| & \leq \|\mathbf{U} - \mathbf{u}\|_{L^3} \|\nabla \mathbf{U}\|_{L^2} \|\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}\|_{L^6} \\
& \leq C \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} \|\nabla \mathbf{U}\|_{L^2} \left( \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} + \|\nabla(\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2} \right), \\
|III| & \leq \|\mathbf{u}\|_{L^3} \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} \|\mathbf{U} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}\|_{L^6} \\
& \leq C \|\nabla \mathbf{u}\|_{L^2} \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} \left( \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} + \|\nabla(\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2} \right), \\
|V| & \leq C \rho^{2m - \frac{n}{2} + 1} \left\{ \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} + \|\nabla(\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2} \right\} \|\mathbf{u}\|_{H^2} \\
|VI| & \leq C \rho^{2m - \frac{n}{2} + 1} \left\{ \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} + \|\nabla(\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2} \right\} \|p\|_{H^1}.
\end{aligned}$$

Terms  $V$  and  $VI$  can be estimated as in the case of the Stokes equations, we refer (46) and (47).

For the pressure term  $IV$ , we apply the same procedure as in the Stokes problem, i.e., use the inf-sup condition which is the hypothesis for the shape functions  $\widehat{\phi}$ 's and  $\psi$ 's. At first, we can obtain the followings

$$\begin{aligned}
IV & = \int_{\Omega} (P - p) \nabla \cdot (\mathbf{U} - \mathbf{u}) \, dx + \int_{\Omega} (P - p) \nabla \cdot (\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, dx \\
& = \int_{\Omega} (\mathcal{S}_{\rho,h}^m p - p) \nabla \cdot (\mathbf{U} - \mathbf{u}) \, dx + \int_{\Omega} (P - p) \nabla \cdot (\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}) \, dx \quad (56) \\
& \leq \|p - \mathcal{S}_{\rho,h}^m p\|_{L^2} \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} \\
& \quad + \left( \|P - \mathcal{S}_{\rho,h}^m p\|_{L^2} + \|p - \mathcal{S}_{\rho,h}^m p\|_{L^2} \right) \|\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}\|_{L^2}.
\end{aligned}$$

We want to find the estimate of  $\|P - \mathcal{S}_{\rho,h}^m p\|_{L^2}$ . As in the case of the Stokes problem, the inf-sup condition implies that

$$\begin{aligned} \lambda \|\mathcal{S}_{\rho,h}^m p - P\|_{L^2} &\leq \frac{\int_{\Omega} (\mathcal{S}_{\rho,h}^m p - P) \nabla \cdot \zeta \, dx}{\|\nabla \zeta\|_{L^2}} \\ &= \frac{1}{\|\nabla \zeta\|_{L^2}} \left[ \nu \int_{\Omega} \nabla(\mathbf{U} - \mathbf{u}) \nabla \zeta \, dx + \int_{\Omega} (\mathbf{U} - \mathbf{u}) \cdot \nabla \mathbf{U} \cdot \zeta \, dx + \int_{\Omega} \mathbf{u} \cdot \nabla(\mathbf{U} - \mathbf{u}) \cdot \zeta \, dx \right. \\ &\quad \left. + \int_{\Omega} (\mathcal{S}_{\rho,h}^m p - p) \nabla \cdot \zeta \, dx - \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \zeta \, d\Gamma + \int_{\partial\Omega} p \zeta \cdot \mathbf{n} \, d\Gamma \right], \end{aligned}$$

for some  $\zeta = \sum_{i=1}^{\widehat{NP}} \mathbf{c}_i \widehat{\phi}_i$ . Thus we have

$$\begin{aligned} \|P - \mathcal{S}_{\rho,h}^m p\|_{L^2} &\leq \frac{\nu + C \|\nabla \mathbf{U}\|_{L^2} + C \|\nabla \mathbf{u}\|_{L^2}}{\lambda} \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} + \frac{1}{\lambda} \|p - \mathcal{S}_{\rho,h}^m p\|_{L^2} \\ &\quad + \frac{C}{\lambda} \rho^{2m - \frac{n}{2} + 1} (\|\mathbf{u}\|_{H^2} + \|p\|_{H^1}). \end{aligned} \quad (57)$$

Thus from the estimates (56) and (57), we obtain

$$\begin{aligned} |IV| &\leq \left( \|p - \mathcal{S}_{\rho,h}^m p\|_{L^2} + \frac{\nu + C \|\nabla \mathbf{U}\|_{L^2} + C \|\nabla \mathbf{u}\|_{L^2}}{\lambda} \|\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}\|_{L^2} \right) \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2} \\ &\quad + \left( 1 + \frac{1}{\lambda} \right) \|p - \mathcal{S}_{\rho,h}^m p\|_{L^2} \|\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u}\|_{L^2} \\ &\quad + C \rho^{2m - \frac{n}{2} + 1} (\|\mathbf{u}\|_{H^2} + \|p\|_{H^1}) \|\nabla(\mathbf{u} - \widehat{\mathcal{R}}_{\rho,h}^m \mathbf{u})\|_{L^2}. \end{aligned}$$

Note that the followings are true

$$\|\nabla \mathbf{U}\|_{L^2} \leq \frac{2}{\nu} \|\mathbf{f}\|_{H^{-1}}, \quad \|\nabla \mathbf{u}\|_{L^2} \leq \frac{2}{\nu} \|\mathbf{f}\|_{H^{-1}}.$$

Combining the estimates of *I*, *II*, *III* and *IV* all together, we finally obtain the following  $L^2$ -error estimates for  $\nabla(\mathbf{U} - \mathbf{u})$  such that

$$\|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2(\Omega)} \leq C_0 \rho^m (\|\mathbf{u}\|_{H^{m+1}} + \|p\|_{H^m}).$$

Then, considering interpolation theorems Theorem 2.1, Corollary 2.1 and  $L^2$  estimate for  $\nabla(\mathbf{U} - \mathbf{u})$ , we obtain the  $L^2$  estimates for  $\mathbf{u}$  and  $p$ . Details of the proof is exactly same as the case of the Stokes equations. Hence we state the stability theorem of the MLSRK scheme for the Navier-Stokes equations.

**THEOREM 3.6.** *Assume  $(\mathbf{u}, p) \in H_0^{m+1}(\Omega) \times H^m(\Omega)$  is the solution of the Navier-Stokes equations (49) and  $(\mathbf{U}, P) \in (C_0^{m+1}(\Omega) \times C^m(\Omega)) \cap ([V_0^h(\Omega)]^n \times M^h(\Omega))$  is the solution of the discrete Navier-Stokes equations (50). For the space dimension  $n = 2$  or  $3$ , if we assume that*

$$\frac{C \|\mathbf{f}\|_{H^{-1}}}{\nu^2} < 1,$$

then the following error estimate is obtained

$$\begin{aligned} \|\mathbf{U} - \mathbf{u}\|_{L^2(\Omega)} + \rho \|\nabla(\mathbf{U} - \mathbf{u})\|_{L^2(\Omega)} + \rho \|P - p\|_{L^2(\Omega)} \\ \leq C_0 \rho^{m+1} (\|\mathbf{u}\|_{H^{m+1}} + \|p\|_{H^m}), \end{aligned}$$

where  $C_0$  depends only on  $\nu$ .

It is interesting that the smallness of  $H^{-1}$  norm of  $\mathbf{f}$  is necessary for the error estimates.

4. Numerical Examples.

Using the MLSRK method discussed in the previous sections, the numerical experiments are performed for the Stokes and the Navier-Stokes equations in two dimensional case. The results show good agreement with stability theorems that we proved.

The domain considered in our examples is the unit square  $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ . The zero boundary conditions for the velocity are imposed on the boundary of  $\Omega$ . Node distributions for the velocity approximation and the pressure approximation are shown in Fig. 1. About the window function, we use the function defined by

$$\Phi(x, y) = S(x) S(y), \tag{58}$$

where  $S(t)$  is the cubic spline function defined below :

$$S(t) = \begin{cases} \frac{2}{3} - 4|t|^2(1 - |t|) & \text{for } |t| \leq \frac{1}{2} \\ \frac{4}{3}(1 - |t|)^3 & \text{for } \frac{1}{2} < |t| < 1 \\ 0 & \text{for } |t| \geq 1. \end{cases}$$

The shape functions of the velocity and the pressure associated with the window function  $\Phi(x, y)$  are drawn in Fig. 2. It turns out that such a distribution of velocity and pressure nodes satisfies the inf-sup condition of the definition 3.

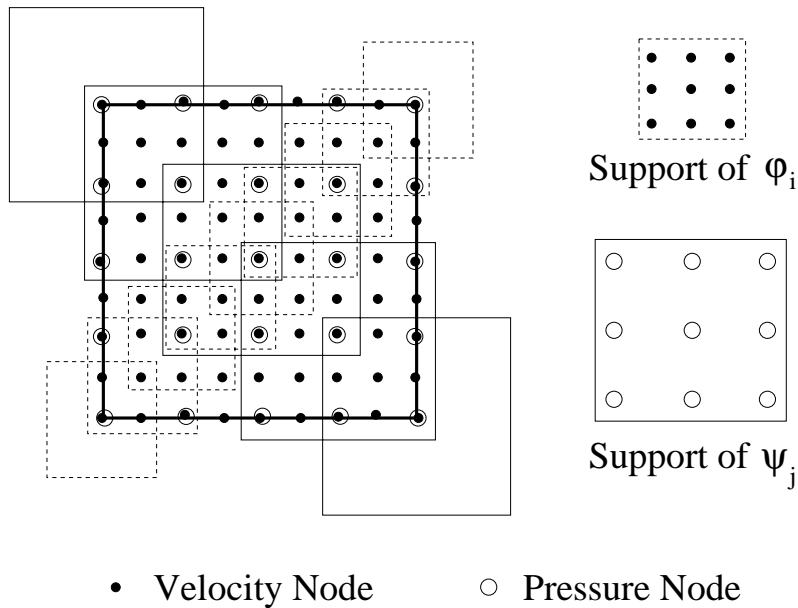


FIGURE 1. Distribution of the velocity and the pressure nodes

In calculating the numerical solution of the Stokes equations, we assume  $\nu = 1$  without loss of generality. In the case of Navier-Stokes equations, we introduce the Reynolds number denoted by  $Re$ . Our numerical experiments for the Navier-Stokes problem are performed when  $Re = 100$ .

To compare the error between the exact solution  $(\mathbf{u}, p)$  and the numerical solution  $(\mathbf{U}, P)$ , we choose the divergence free velocity  $\mathbf{u} = (u, v)$  and the pressure  $p$  in

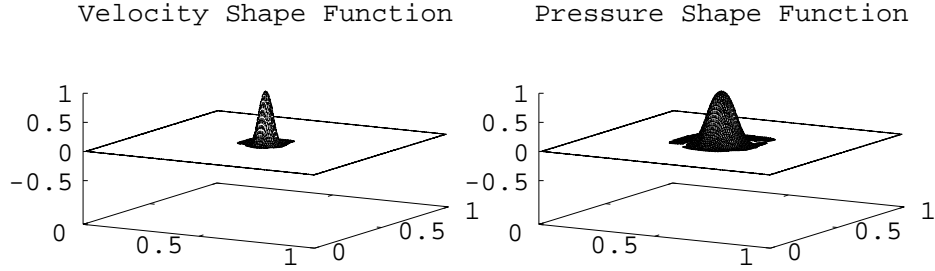


FIGURE 2. Velocity Shape Function(left) and Pressure Shape Function(right)

advance such as

$$\begin{aligned} u &= \pi \sin^3 \pi x \sin^2 \pi y \cos \pi y, \\ v &= -\pi \sin^2 \pi x \sin^3 \pi y \cos \pi x, \\ p &= x^2 - y^2. \end{aligned}$$

Then its corresponding force  $\mathbf{f}$  can be exactly calculated for each case of the Stokes and the Navier-Stokes equations. Under this situation, we made the numerical solutions  $\mathbf{U}$  and  $P$  of the Stokes flow and the Navier-Stokes flow problems using MLSRK scheme.

Fig. 3 shows the logarithmic scale plots of decay rates of relative errors for the Stokes problem and the navier-Stokes problem. As a reference, relative  $L^2$ -,  $H^1$  errors for  $\mathbf{U} = (U, V)$  and relative  $L^2$ - error for  $P$  are tabulated in Table 1, Table 2 and Table 3 respectively, where the letter  $m$  denotes the order of consistency.

Nodes Vel. $\times$ Pres.	Stokes		Navier-Stokes( $Re = 100$ )	
	m=1	m=2	m=1	m=2
441 $\times$ 121	1.49440945e-02	1.26810795e-03	1.48085601e-02	1.26768178e-03
1681 $\times$ 441	3.74951155e-03	1.49084512e-04	3.73612781e-03	1.49022236e-04
3721 $\times$ 961	1.66755670e-03	4.91346047e-05	1.66174975e-03	4.90983793e-05
6561 $\times$ 1681	9.38218487e-04	2.38447287e-05	9.34962190e-04	2.38231316e-05

TABLE 1. Relative  $L^2$ - errors of  $\mathbf{U} - \mathbf{u}$ 

Nodes Vel. $\times$ Pres.	Stokes		Navier-Stokes( $Re = 100$ )	
	m=1	m=2	m=1	m=2
441 $\times$ 121	1.23078917e-01	2.06432348e-02	1.23092386e-01	2.06434938e-02
1681 $\times$ 441	6.16188431e-02	5.13782186e-03	6.16200359e-02	5.13783366e-03
3721 $\times$ 961	4.10890061e-02	2.28167390e-03	4.10893707e-02	2.28167686e-03
6561 $\times$ 1681	3.08193201e-02	1.28287371e-03	3.08194759e-02	1.28287502e-03

TABLE 2. Relative  $H^1$ - errors of  $\mathbf{U} - \mathbf{u}$ 

Though we discussed the problems only with the zero velocity condition on the boundary of  $\Omega$  in this paper, the generalization for the problem of non-zero boundary condition is readily obtained. As an example of the problem of non-zero boundary condition, the driven cavity flow is calculated numerically. We assume

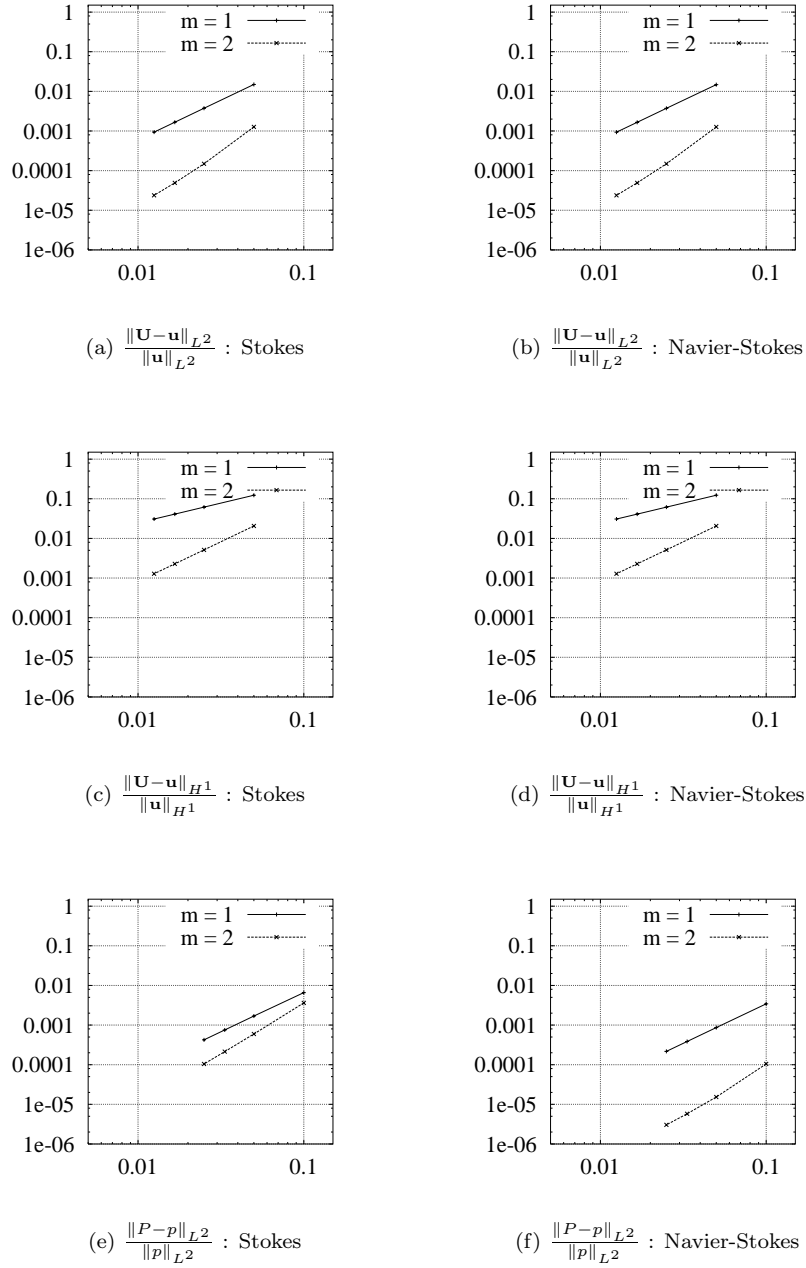


FIGURE 3. Decay rates of relative errors

the domain of flow is the same  $\Omega$  as previous example and the upper wall ( $0 \leq x \leq 1, y = 1$ ) moves to right with unit speed while the other walls remain fixed. To impose boundary condition, we made *boundary transformation* for the shape function in the numerical code as we did for the previous example. In numerical

Nodes Vel. $\times$ Pres.	Stokes		Navier-Stokes( $Re = 100$ )	
	m=1	m=2	m=1	m=2
441 $\times$ 121	6.58106632e-03	3.63083620e-03	3.43864089e-03	1.04801197e-04
1681 $\times$ 441	1.68942766e-03	5.94784716e-04	8.66369582e-04	1.52837464e-05
3721 $\times$ 961	7.53849631e-04	2.14239418e-04	3.85350328e-04	5.79634323e-06
6561 $\times$ 1681	4.24609588e-04	1.04359181e-04	2.16815953e-04	3.04217952e-06

TABLE 3. Relative  $L^2$ - errors of  $P - p$ 

experiment, we put the Reynolds number  $Re = 1$ ,  $Re = 100$  for the Stokes problem and the Navier-Stokes problem. The stream lines and the pressure are shown in Fig. 4, respectively. Here the number of velocity nodes is 6561 and that of pressure nodes is 1681.

In Fig. 4, the symmetry of our numerical solution is well presented, which is intrinsic property of the Stokes flow in a symmetric domain. The skewness of the solution for the Navier-Stokes flow is also well illustrated.

**5. Conclusions.** The meshless method for the Stokes and the Navier-Stokes problems are analyzed rigorously and we have performed several numerical experiments successfully. The numerical results show a good agreement with the theoretic analysis for the convergence of the discrete solution. The MLSRK method seems recommendable for the incompressible Navier-Stokes flow and the Stokes flow. In the MLSRK method for the incompressible viscous flow problems, only if we distribute the nodes in the computational domain, we can calculate the numerical solution as smooth as we want. Furthermore, this method has the advantage of mesh adaptation without doubt. Therefore, we may conclude that the MLSRK must be one of the promising methods recently discovered and much more intense researches are required for the more general applications to various directions.

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## REFERENCES

- [1] R. A. Gingold and J.J. Monaghan, SMOOTHED PARTICLE HYDRODYNAMICS : THEORY AND APPLICATION TO NON-SPHERICAL STARS, Mon. Not. R. Astron. Soc., Vol. 181 (1977), 275–389.
- [2] S. Li and W. K. Liu, REPRODUCING KERNEL HIERARCHICAL PARTITION OF UNITY, PART I – FORMULATION AND THEORY, Int. J. Numer. Meth. Engng., Vol. 45 (1999), 251–288.
- [3] S. Li and W. K. Liu, REPRODUCING KERNEL HIERARCHICAL PARTITION OF UNITY, PART II – APPLICATIONS, Int. J. Numer. Meth. Engng., Vol. 45 (1999), 289–317.
- [4] L.D. Libersky, A. G. Petschek, T. C. Carney, J. R. Hipp and F. A. Allahdadi, HIGH STRAIN LAGRANGIAN HYDRODYNAMICS – A THREE DIMENSIONAL SPH CODE FOR DYNAMIC MATERIAL RESPONSE, J. Comput. Phys., Vol. 109 (1993), 67–75.
- [5] W. K. Liu, S. Jun, S. Li, J. Adee and T. Belytschko, REPRODUCING KERNEL PARTICLE METHODS FOR STRUCTURAL DYNAMICS, Int. J. Numer. Meth. Engng., Vol. 38 (1995), 1655–1679.
- [6] W. K. Liu, S. Jun and Y. F. Zhang, REPRODUCING KERNEL PARTICLE METHODS, Int. j. numer. methods fluids, Vol. 20 (1995), 1081–1106.
- [7] W. K. Liu and Y. Chen, WAVELET AND MULTIPLE SCALE REPRODUCING KERNEL METHODS, Int. j. numer. methods fluids, Vol. 21 (1995), 901–931.

- [8] W. K. Liu, S. Li and T. Belytschko, MOVING LEAST SQUARE REPRODUCING KERNEL METHODS (I) METHODOLOGY AND CONVERGENCE, *Comput. Methods Appl. Mech. Engng.*, Vol. 143 (1996), 422–433.
- [9] Y. Y. Lu, T. Belytschko and L. Gu, A NEW IMPLEMENTATION OF THE ELEMENT FREE GALERKIN METHOD, *Comput. Methods Appl. Mech. Engng.*, Vol. 113 (1994), 397–414.
- [10] J.M. Melenk and I. Babuška, THE PARTITION OF UNITY FINITE ELEMENT METHOD : BASIC THEORY AND APPLICATIONS, *Comput. Methods Appl. Mech. Engng.*, Vol. 139 (1996), 289–314.
- [11] B. Nayroles, G. Touzot and P. Villon, GENERALIZING THE FINITE ELEMENT METHOD : DIFFUSE APPROXIMATION AND DIFFUSE ELEMENTS, *Comput. Mech.*, Vol. 10 (1992), 307–318.

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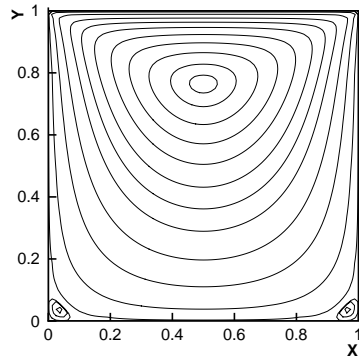
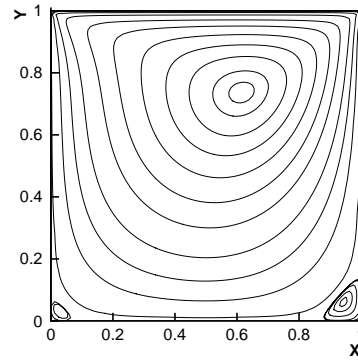
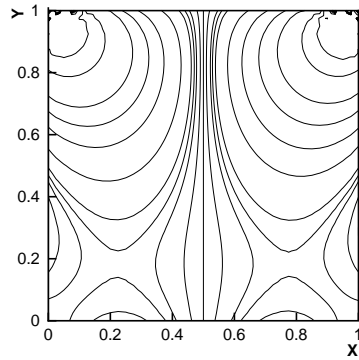
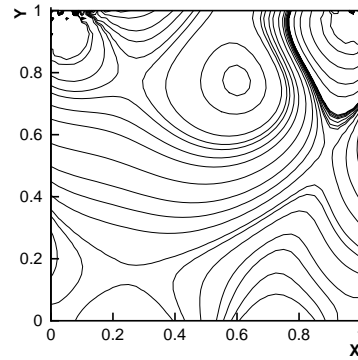
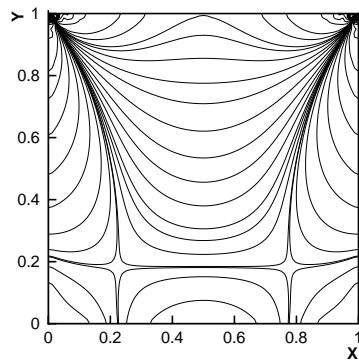
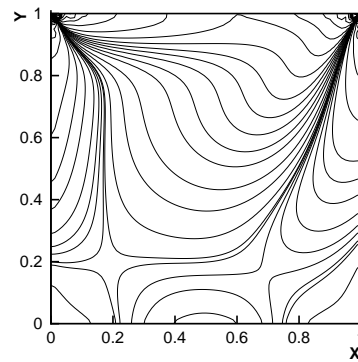
(a) Stream lines : Stokes( $Re = 1$ )(b) Stream lines : Navier-Stokes( $Re = 100$ )(c) Pressure : Stokes( $Re = 1$ )(d) Pressure : Navier-Stokes( $Re = 100$ )(e) Vorticity : Stokes( $Re = 1$ )(f) Vorticity : Navier-Stokes( $Re = 100$ )

FIGURE 4. Stokes(a,c,e) and Navier-Stokes(b,d,f) driven cavity flows