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Finite element approximations for the Stokes equations on curved domains, and their errors ☆

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Abstract

We construct finite element spaces for the Stokes equations on curved domains, which satisfy the Brezzi–Babuška condition. Moreover, we estimate the errors of the finite element approximations for the Stokes equations.

Keywords: Brezzi–Babuška condition; FEM; Stokes equations; Errors

1. Introduction and statement of the result

Many numerical analysts have studied finite element approximations for the Stokes equations and their errors. For error estimates, it is assumed that the solution of Stokes equations are sufficiently regular, see [7]. However, if the domain of Stokes flow is not smooth enough, in other words, if the boundary of the domain has a geometric singularity, then the regularity of the solutions is not known yet in general. A polygonal domain is a typical example of a singular domain. Only on a few cases of polygonal domains the regularity is known. For example, Kellog and Osborn [8] have shown that on a con-

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vex polygon in $\mathbb{R}^2$ the solution of the Stokes equations is regular. Since the regularity of solutions of the Stokes equations is not known (even though many people want to believe the regularity) on general polygonal domains, many numerical analysts assume that the solutions are regular on polygonal domains. However, in this paper we do not assume artificially the regularity. Instead we use the well-known regularity of solutions on smooth domains. We approximate the smooth domain with polygons, and then estimate the errors of finite element approximations on the polygons.

Historically, there are many papers which was dealt with finite element approximations on curved domains. Schatz and Wahlbin [10] showed that the H_0^1 projection into finite element spaces based on quasi-uniform triangulations of a bounded smooth domain in $\mathbb{R}^n$ is stable in the maximum norm. Ciarlet and Raviart [5], Lenoir [9] considered interpolations and finite elements, and Bernardi [2] constructed L^2-interpolations into finite element spaces on curved domains, and estimated the errors. Schatz et al. [11] considered finite element approximations on convex domains for the heat equations.

We consider a smooth domain, and approximate the domain by polygonal subdomains. On the subdomains we construct finite element spaces and we approximate the solution. Then, we show that the errors of the approximations in H^1-norm for the velocity and in L^2-norm for the pressure of the Stokes equations are $\mathrm{O}(h)$, and that the L^2-error of the velocity is $\mathrm{O}(h^2)$, where h is a discretization parameter tending to zero.

Let Ω be a bounded domain in $\mathbb{R}^2$ of which boundary $\partial\Omega$ is in $\mathscr{C}^2$. We consider the stationary Stokes equations: for $\mathbf{f}$ given in $L^2(\Omega)^2$, find $(\mathbf{u}, p)$ in $H_0^1(\Omega)^2 \times L_a^2(\Omega)$ such that

$$\begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}(\mathbf{x}) = 0 & \text{for } \mathbf{x} \in \partial\Omega, \end{cases} \tag{1.1}$$

where $L_a^2(\Omega)$ is the subspace of $L^2(\Omega)$ with $\int_\Omega q\,\mathrm{d}\mathbf{x} = 0$ for all $q \in L_a^2(\Omega)$. We denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\Omega)$ or in $L^2(\Omega)^2$, by $\|\cdot\|$ the corresponding norm obtained by the inner product $\langle \cdot, \cdot \rangle$. For short, set $X \overset{\text{def}}{=} H_0^1(\Omega)^2$ and $Q \overset{\text{def}}{=} L_a^2(\Omega)$.

Let h denote a discretization parameter tending to zero. For each h, let X_h and Q_h be finite dimensional spaces such that

$$X_h \subset X = H_0^1(\Omega)^2, \quad Q_h \subset Q = L_a^2(\Omega).$$

We consider the discretized problem: find $(\mathbf{u}_h, p_h)$ in $X_h \times Q_h$ such that

$$\begin{cases} \langle \nabla\mathbf{u}_h, \nabla\mathbf{v}_h \rangle - \langle p_h, \operatorname{div}\mathbf{v}_h \rangle = \langle \mathbf{f}, \mathbf{v}_h \rangle \text{ for all } \mathbf{v}_h \in X_h, \\ \langle q_h, \operatorname{div}\mathbf{u}_h \rangle = 0 \text{ for all } q_h \in Q_h. \end{cases} \tag{1.2}$$

For the existence and the uniqueness of solution $(\mathbf{u}_h, p_h)$ of (1.2), a sufficient condition is known, which is called the discrete *inf-sup*, or the Brezzi–Babuška(LBB) condition:

there exists a constant $\beta > 0$ independent of h such that

$$\sup_{\mathbf{v}\in X_h} \frac{\langle \mathbf{v}_h, q_h\rangle}{\|\mathbf{v}_h\|_{X_h}} \geqslant \beta\|q_h\| \quad \text{for all } q_h \in Q_h. \tag{1.3}$$

It is also known that the necessary and sufficient condition of the LBB condition is that

there exists a linear operator $\Pi_h \in \mathscr{L}(X; X_h)$ satisfying:

$$\begin{aligned} &\langle \operatorname{div}(\mathbf{v} - \Pi_h\mathbf{v}), q_h\rangle = 0 \quad \text{for all } q_h \in Q_h,\ \mathbf{v} \in X, \\ &\|\Pi_h\mathbf{v}\|_{X_h} \leqslant C\|\mathbf{v}\|_X \quad \text{for all } \mathbf{v} \in X, \end{aligned} \tag{1.4}$$

with a constant $C > 0$ independent of h.

In the case of a polygonal domain $\Omega = \Omega_h$, the finite element spaces satisfying the Brezzi–Babuška condition, are constructed by many authors, see [6] for the reference.

In this paper we construct some finite element spaces on curved domains satisfying the Brezzi–Babuška condition, and we estimate the errors of finite element approximations for the Stokes equations. In Section 2 we review finite elements. Since functions in $H_0^1(\Omega)$ may not be continuous, direct construction of finite element spaces satisfying the inf-sup condition is not easy. So we consider continuous L^2-interpolations in Section 3. Based on the interpolations we construct finite element spaces satisfying the inf-sup condition in Section 4. In Section 5, we estimate the errors of finite element approximations for the Stokes equations.

2. Review of finite element spaces

In this section we review Lagrange finite elements. Before going further, for our usage we recall the regularity result of Stokes equations given in Proposition 2.3 [12]:

Proposition 2.1. *Let Ω be an open bounded set of class $\mathscr{C}^r$, $r = \max\{m+2, 2\}$, m integer $\geqslant -1$. If $\mathbf{f} \in W^{m,s}(\Omega)$ with $1 < s < \infty$, then there exist a unique solution $(\mathbf{u}, p)$ of* (1.1) (*p is unique up to a constant*):

$$\mathbf{u} \in W^{m+2,s}(\Omega)^2, \quad p \in W^{m+1,s}(\Omega),$$

and there exists a constant $c_0(s, m, \Omega)$ such that

$$\|\mathbf{u}\|_{W^{m+2,s}(\Omega)} + \|p\|_{W^{m+1,s}(\Omega)/\mathbb{R}} \leqslant c_0 \|\mathbf{f}\|_{W^{m,s}(\Omega)}.$$

Remark. If m is sufficiently large, for example $s_1 s > 2$ and $m+2 = s_1 + s_2$, then the solution $\mathbf{u} \in \mathscr{C}^{s_2}(\Omega)$ by Sobolev embedding Theorem, see [1].

For $K \subset \Omega$, integer $i \geqslant 0, W^{i,s}(K)$ and $\|\cdot\|_{i,s,K}$ denote the usual Sobolev space and its norm on K, respectively. For $K \subset \Omega$, integer $i \geqslant 0$, let $|\mathbf{v}|_{i,K}$ be the $L^2(K)$-norm of ith order derivatives of $\mathbf{v}$ on K, which is the semi-norm of $H^i(K)^2$. For $s = 2$, set $\|\mathbf{v}\|_{i,K} \stackrel{\text{def}}{=} \|\mathbf{v}\|_{i,2,K}$, and $H^i(K) = W^{i,2}(K)$ for $K \subset \mathbb{R}^2$. For $s = 2$ and $i = 0$, we set $\|\mathbf{v}\|_K \stackrel{\text{def}}{=} \|\mathbf{v}\|_{0,K}$ for $K \subset \mathbb{R}^2$. If V is a function space, we also denote by $\|\mathbf{v}\|_V$ the norm of $\mathbf{v} \in V$. For the definitions of Sobolev spaces, refer to Adams [1].

We now review Lagrange finite elements. We assume that Ω is open bounded domain in $\mathbb{R}^2$ of which boundary $\partial\Omega$ is Lipschitz or smooth. For our error estimate we will assume later that Ω is of class $\mathscr{C}^2$. For each $h > 0$, we introduce a triangulation $\mathscr{T}_h$ of Ω, i.e., a finite set of 2-simplices $K \subset \Omega$, where h denotes the maximal diameter of K in $\mathscr{T}_h$; set

$$\Omega_h \stackrel{\text{def}}{=} \cup_{K \in \mathscr{T}_h} K.$$

In the triangulation $\mathscr{T}_h$ any intersection of two 2-simplices is empty or a vertex, or a 1-face. It is clear that $\Omega_h \neq \Omega$ in general, that is, $\mathscr{T}_h$ is not exact. We notice that in general, when Ω_h is a polygonal domain, $\Omega_h \not\subset \Omega$, unless (i) Ω is a polygon, or (ii) Ω is convex smooth curve and the partitions of Ω_h are straight edged, or (iii) $\partial\Omega$ is a polynomial curve and isoparametric finite elements are used at the boundary. Here, our polygon Ω_h is contained in Ω.

For each $\mathbf{x} \in \partial\Omega_h$, we define $\delta(\Omega, \mathbf{x})$, and then $\delta(\Omega, \Omega_h)$ by

$$\delta(\Omega, \mathbf{x}) \stackrel{\text{def}}{=} \operatorname{dist}(\mathbf{x}, \partial\Omega), \quad \text{and} \quad \delta(\Omega, \Omega_h) \stackrel{\text{def}}{=} \sup_{\mathbf{x} \in \partial\Omega_h} \delta(\Omega, \mathbf{x}). \tag{2.1}$$

The boundary $\partial\Omega_h$ of Ω_h consists of vertices and 1-faces. We assume that $\Omega_h \subset \Omega, \Omega$ is in $\mathscr{C}^2$ and $\delta(\Omega, \Omega_h) \leqslant Ch^2$.

For each $K \in \mathscr{T}_h$, there is an affine mapping $F_K : \mathbb{R}^2 \to \mathbb{R}^2$ which maps a 2-simplex $\widehat{K}$ onto K and which is of the form

$$F_K(\hat{\mathbf{x}}) = B_K \hat{\mathbf{x}} + b_K,$$

where B_K is a linear mapping from $\mathbb{R}^2$ to itself and $b_K \in \mathbb{R}^2$. The 2-simplex $\widehat{K}$ is called a reference 2-simplex. For each h and for each $K = F_K(\widehat{K})$ in $\mathscr{T}_h$, we denote by h_K the diameter of K and by ρ_K the maximal diameter of circles inscribed in K. We notice that

$$\|B_K\| \leqslant h_K / \hat{\rho}_{\hat{K}}, \quad \|B_K^{-1}\| \leqslant \hat{h}_{\hat{K}} / \rho_K,$$

where $\hat{h}_{\hat{K}}$ is the diameter of $\widehat{K}$, and $\hat{\rho}_{\hat{K}}$ the maximal diameter of circles inscribed in $\widehat{K}$. The family $\{\mathscr{T}_h\}_h$ of triangulations is said to be *regular* if there exist a constant c independent of h such that

$$h_K/\rho_K \leqslant c$$

for any element K in $\mathscr{T}_h$. In [6], the following is shown:

Lemma 2.2. *For each $m \geqslant 0$ and for all real s with $1 \leqslant s \leqslant \infty$, the mapping $\mathbf{v} \mapsto \hat{\mathbf{v}} = \mathbf{v} \circ F_K$ is an isomorphism from $W^{m,s}(K)^2$ onto $W^{m,s}(\widehat{K})^2$ and the following bounds hold*:

$$|\mathbf{v}|_{m,s,K} \leqslant C|\det B_K|^{1/s} \|B_K^{-1}\|^m |\hat{\mathbf{v}}|_{m,s,\hat{K}} \text{ for all } \hat{\mathbf{v}} \in W^{m,s}(\widehat{K})^2,$$
$$|\hat{\mathbf{v}}|_{m,s,\hat{K}} \leqslant C|\det B_K|^{-1/s} \|B_K\|^m |\mathbf{v}|_{m,s,K} \text{ for all } \mathbf{v} \in W^{m,s}(K)^2.$$

For a given integer $r-1 \geqslant k \geqslant 1$ and a 2-simplex K, denote by $P_k(K)$ the set of all polynomials of degree $\leqslant k$, and set $P_k(K) = P_k(K)^2$. Notice that any polynomial $\rho \in P_k(K)$ is uniquely determined by its values on the set

$$S_k(K) = \left\{ \mathbf{x} = \sum_{j=1}^{3} \lambda_j \mathbf{a}_j : \sum_{j=1}^{3} \lambda_j = 1, \quad \lambda_j \in \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\},\ 1 \leqslant j \leqslant 3 \right\},$$

where $\mathbf{a}_j$ are the vertices of K. For each $K \in \mathscr{T}_h$, $(K, \mathbf{P}_k(K), S_k(K))$ is a Lagrange finite element if we identify $\mathbf{x}_i \in S_k(K)$ with $\delta_{\mathbf{x}_i}$, where $\delta_{\mathbf{x}_i}$ is the Dirac delta function at $\mathbf{x}_i$.

For a 2-simplex K with vertices $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$, the barycentric coordinates $\lambda_i(\mathbf{x}), = 1, 2, 3$, of any point $\mathbf{x} \in K$, are the solutions of the linear system

$$\sum_{i=1}^{3} a_{ij}\lambda_i = x_j, \quad j = 1, 2, \quad \text{and} \quad \sum_{i=1}^{3} \lambda_i = 1,$$

where $\mathbf{a}_i = (a_{i1}, a_{i2})$, and $\mathbf{x} = (x_1, x_2)$. Let $\hat{\lambda}_i = \lambda_i \circ F_K$ be the barycentric coordinates on $\widehat{K}$.

3. Continuous L^2-interpolations

Since each function $\mathbf{v}$ in X to be interpolated may not be continuous, the usual interpolation $\mathscr{I}_h\mathbf{v} \in P_k(K)$ defined in each element K by:

$$\mathscr{I}_h\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) \quad \text{for } \mathbf{x} \in S_k(K)$$

may not be defined. So we cannot define their interpolations directly by using the linear forms that characterize the degrees of freedom of the finite elements. To overcome this difficulty, we introduce an interpolation operator by using a local L^2-projection as in [2]. Based on the interpolation we finally want to define a projection $\Pi_h: X \to X_h$ which satisfies the inf-sup condition of the finite element space.

First, we consider local L^2-interpolations. Define finite dimensional subspace Y_h of $H^1(\Omega)^2$ by

$$Y_h \stackrel{\text{def}}{=} \{\rho \in \mathscr{C}^0(\Omega_h) : \rho|_K \in \mathbf{P}_k(K) \quad \text{for all } K \in \mathscr{T}_h\}.$$

We take nodes $\mathbf{x}_i \in \Omega_h$ such that

$$\cup_{K\in\mathscr{T}_h} S_k(K) = \{\mathbf{x}_i : 1 \leqslant i \leqslant N_h\},$$

all the nodes $\mathbf{x}_i$ being distinct. For each integer i, $1 \leqslant i \leqslant N_h$, there exists exactly one function $\varphi_i \in Y_h$ such that for $j = 1, \ldots, N_h$,

$$\begin{cases} \varphi_i(\mathbf{x}_j) = 1 & \text{if } i = j, \\ \varphi_i(\mathbf{x}_j) = 0 & \text{if } i \neq j. \end{cases}$$

Then the set $\{\varphi_i: 1 \leqslant i \leqslant N_h\}$ is a basis of Y_h. Then, for $1 \leqslant i \leqslant N_h$, we define a macroelement Δ_i by

$$\Delta_i = \cup\{K \in \mathscr{T}_h;\ \text{supp } (\varphi_i) \cap K \text{ is nonempty}\} = \cup\{K \in \mathscr{T}_h : \mathbf{x}_i \in K\}.$$

If the triangulation $\mathscr{T}_h$ is regular, then the number of triangles K in Δ_i is bounded by a constant M independent of h and i. On the other hand, the number of macroelements Δ_i containing a given K is bounded by a constant. To each Δ_i, there exists a reference macroelement $\widehat{\Delta}_i$ contained in the unit ball of $\mathbb{R}^2$ which is the union of $\widehat{K} = F_K^{-1}(K), K \subset \Delta_i$. Furthermore, there exists a continuous and invertible mapping F_{Δ_i} from $\widehat{\Delta}_i$ onto Δ_i such that $F_{\Delta_i}|_{\hat{K}}$ is an affine mapping from $\widehat{K}$ onto $K = F_K(\widehat{K})$ for each $\widehat{K} \subset \widehat{\Delta}_i$. Some examples of Δ_i and $\widehat{\Delta}_i$ are given in Fig. 1.

Now we define the interpolation operator $\mathscr{R}_h^0: \mathbf{L}_0^1(\Omega) \to Y_h^0$ by using local L^2-projections, where $\mathbf{L}_0^1(\Omega)$ is the subset of $L^1(\Omega)^2$ with zero boundary data, and

$$Y_h^0 \stackrel{\text{def}}{=} Y_h \cap H_0^1(\Omega_h)^2$$

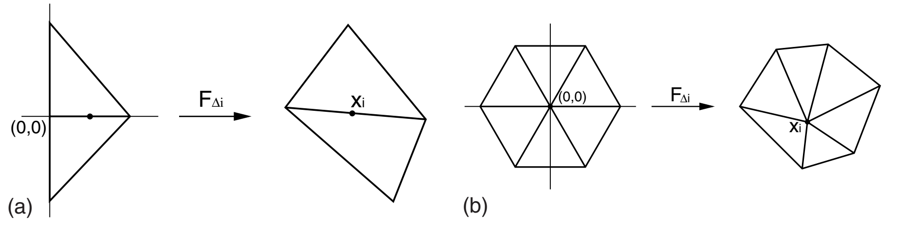

Fig. 1. Examples of Δ_i and their references $\widehat{\Delta}_i$.

is the subset of Y_h with zero boundary data. Let Γ_h be the set of the 1-faces $f \subset \partial\Omega_h$ of $K \in \mathscr{T}_h$. For each 1-face $f \in \Gamma_h$ of $K \in \mathscr{T}_h$, denote

$$S_{h,\Gamma_h} = \cup_{f\in\Gamma_h}(S_k(K) \cap f).$$

We extract a maximal system of linearly independent functions from $\{\varphi_i: i = 1, \dots, N_h\}$ corresponding to S_{h,Γ_h}, which is denoted by $\{\varphi_1, \dots, \varphi_{N'_h}\}$. Then we complete the system in Y_h by renumbering to obtain the maximal system $\{\varphi_1, \dots, \varphi_{N_h}\}$. We have the finite element space Y_h^0 with zero boundary condition;

$$Y_h^0 = \mathrm{span}\{\varphi_i : N'_h < i \leqslant N_h\}.$$

Now we are ready to define the interpolation operator $\mathscr{R}_h^0$.

Let $\mathbf{v} \in \mathbf{L}_0^1(\Omega)$. For i, $1 \leqslant i \leqslant N_h$, setting $\hat{\mathbf{v}}_i \stackrel{\mathrm{def}}{=} \mathbf{v} \circ F_{\Delta_i}$, there is a unique $\hat{\rho}_i \in \mathbf{f}P_k(\widehat{\Delta}_i)$ such that

$$\int_{\widehat{\Delta}_i} (\hat{\mathbf{v}}_i - \hat{\rho}_i)\hat{\rho}\, \mathrm{d}\hat{\mathbf{x}} = 0 \quad \text{for all } \hat{\rho} \in \mathbf{P}_k(\widehat{\Delta}_i), \tag{3.1}$$

where $\mathbf{P}_k(\widehat{\Delta}_i)$ is the subset of $\mathscr{C}^0(\widehat{\Delta}_i)$ of piecewise polynomials of degree $\leqslant k$. Then, we define a local interpolation $\mathscr{R}_h\mathbf{v}$ by

$$\mathscr{R}_h\mathbf{v} \stackrel{\mathrm{def}}{=} \sum_{i=1}^{N_h} \left(\hat{\rho}_i \circ F_{\Delta_i}^{-1}\right)(\mathbf{x}_i)\varphi_i. \tag{3.2}$$

The interpolation $\mathscr{R}_h\mathbf{v}$ belongs to Y_h. In the definition of the interpolation $\mathscr{R}_h\mathbf{v}$, the boundary information of $\mathbf{v}$ is not counted on, in other words, $\mathscr{R}_h\mathbf{v}$ may not belong to Y_h^0. To get an interpolation in Y_h^0 for $\mathbf{v} \in \mathbf{L}_0^1(\Omega)$, we need to define another interpolation, denoted by $\mathscr{R}_h^0$. If the points $\mathbf{x}_i$ of Δ_i is on $\partial\Omega_h$, then discard such i from (3.2). That is, we discard the indices $i = 1, \dots, N'_h$. Then we define $\mathscr{R}_h^0\mathbf{v}$ by

$$\mathscr{R}_h^0\mathbf{v} \stackrel{\mathrm{def}}{=} \sum_{i=N'_h+1}^{N_h} \left(\hat{\rho}_i \circ F_{\Delta_i}^{-1}\right)(\mathbf{x}_i)\varphi_i. \tag{3.3}$$

The interpolation $\mathscr{R}_h^0\mathbf{v}$ for $\mathbf{v} \in \mathbf{L}_0^1(\Omega)$ belongs to Y_h^0.

Remark. If $\mathbf{v} \in W^{2,s}(\Omega)^2$ for $s > 1$, then the usual interpolation $\mathscr{I}_h\mathbf{v}$ is defined, and we use that instead of $\mathscr{R}_h\mathbf{v}$. In the same way in (3.3), the interpolation $\mathscr{I}_h^0\mathbf{v}$ with zero on the boundary is defined from $\mathscr{I}_h\mathbf{v}$.

The following is shown in Lemma 4.1 [2].

Lemma 3.1. *Assume that the triangulation $\mathscr{T}_h$ is regular. For any function $\mathbf{v}$ in $\mathbf{L}^1(\Omega) \cap W^{\ell,s}(\Delta_i)^2$, we have that, for each K of $\mathscr{T}_h$ contained in Δ_i,*

$$\|\mathbf{v} - \rho_i\|_{m,r,K} \leqslant Ch^{\ell-m+2(1/r-1/s)} \|\mathbf{v}\|_{\ell,s,\Delta_i}$$

for $m \leqslant \ell \leqslant k+1$, *where* $\hat{\rho}_i = \rho_i \circ F_{\Delta_i}$.

The error estimate of the L^2 projection $\mathscr{R}_h$ is given in [2].

Lemma 3.2. *Assume that the triangulation* $\mathscr{T}_h$ *is regular. Given* $K \in \mathscr{T}_h$, *set* Δ_K *be the union of all* Δ_i *containing* $K, i \leqslant 1 \leqslant N_h$. *Then, for any function* $\mathbf{v}$ *in* $\mathbf{L}^1(\Omega) \cap W_0^{\ell,s}(\Delta_K)^2$, *there is a constant C independent of h such that*

$$\|\mathbf{v} - \mathscr{R}_h\mathbf{v}\|_{m,r,K} \leqslant Ch^{\ell-m+2(1/r-1/s)} \|\mathbf{v}\|_{\ell,s,\Delta_K},$$

for $m \leqslant \ell \leqslant k+1$.

The corresponding lemma for $\mathscr{I}_h$ to Lemma 3.2 is also given in Lemma I.A.2 [6].

If the triangulation $\mathscr{T}_h$ is exact, that is, $\Omega_h = \Omega$ then the error estimate of $\mathscr{R}_h^0$ is given in [2]. In our case, the triangulation $\mathscr{T}_h$ may not be exact. Before we estimate the errors of the projection $\mathscr{R}_h^0$, we need the following lemma:

Lemma 3.3 (Poincaré's inequality). *Let* $K \in \mathscr{T}_h$ *be a triangle sharing its one face with* $\partial\Omega_h$. *Assume that* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ *be the vertices of K and that* $\mathbf{a}_1, \mathbf{a}_3$ *are on* $\partial\Omega_h$. *Let* $\mathbf{b}_1, \mathbf{b}_3$ *be the two closest points on* $\partial\Omega$ *from* $\mathbf{a}_1, \mathbf{a}_3$, *respectively. Set* $\widetilde{K}$ *be a subset of* Ω *containing K, like in Fig. 2, which is surrounded by the other two sides of K, the two line segments* $\overline{\mathbf{a}_1\mathbf{b}_1}$ *and* $\overline{\mathbf{a}_3\mathbf{b}_3}$, *and the curve* $\gamma \stackrel{\text{def}}{=} \widehat{\mathbf{b}_1\mathbf{b}_3}$ *on* $\partial\Omega$. *Then, for* $\mathbf{v} \in H^1(\widetilde{K})^2$ *with* $\mathbf{v}|_\gamma = 0$, *we have*

$$\|\mathbf{v}\|_K \leqslant (\delta(\Omega, \Omega_h) + h)\|\nabla\mathbf{v}\|_{\widetilde{K}},$$

where $\delta(\Omega, \Omega_h)$ *is defined in* (2.1).

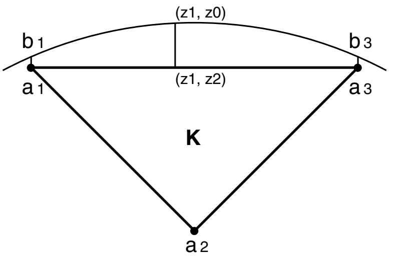

Fig. 2. Triangle on $\partial\Omega_h$.

We estimate the errors of the L^2-projection $\mathscr{R}_h^0$.

Theorem 3.4. *Assume that the triangulation $\mathscr{T}_h$ is regular, and $\delta(\Omega, \Omega_h) \leqslant Ch^2$. Given $K \in \mathscr{T}_h$, let Δ_K be the union of all Δ_i containing K, $i \leqslant 1 \leqslant N_h$. If $K \in \mathscr{T}_h$ has a nonempty intersection with $\partial\Omega_h$, then we denote by $\widetilde{\Delta}_K$*

$$\widetilde{\Delta}_K \stackrel{\text{def}}{=} \Delta_K \cup \bigcup_{G \subset \Delta_K, G \in \mathscr{T}_h} \widetilde{G},$$

where $\widetilde{G}$ is defined like $\widetilde{K}$ in Lemma 3.3. Then, for any function $\mathbf{v}$ in $\mathbf{L}_0^1(\Omega) \cap H_0^1(\Delta_K)^2$, there is a constant C independent of h such that

$$\|\mathbf{v} - \mathscr{R}_h^0\mathbf{v}\|_{m,K} \leqslant Ch^{1-m}\|\mathbf{v}\|_{1,\tilde{\Delta}_K}.$$

where $m = 0, 1$.

Furthermore, if $s > 2$, then we have that for $0 \leqslant m \leqslant 2$,

$$\|\mathbf{v} - \mathscr{I}_h^0\mathbf{v}\|_{m,s,K} \leqslant Ch^{2-m}\|\mathbf{v}\|_{2,s,\tilde{\Delta}_K} \quad \textit{for } \mathbf{v} \in W_0^{2,s}(\Omega)^2.$$

Remark. If K has no intersection with $\partial\Omega_h$ then $\widetilde{\Delta}_K = \Delta_K$.

Proof. If K is in the interior of Ω_h then $\mathscr{R}_h^0$ is $\mathscr{R}_h$. Hence, the proof follows from Lemma 3.2. We assume that K has a face or a vertex on $\partial\Omega_h$. Then we renumber the macroelements $\Delta_i, i = 1, \ldots, j$ such that $K \subset \Delta_i$. Among them, we denote by $\Delta_1, \ldots, \Delta_{j'}, j' < j$, the macroelements such that the node $\mathbf{x}_i$ corresponding to Δ_i belongs to S_{h,Γ_h} for $i = 1, \ldots, j'$. We have

$$\mathbf{v} - \mathscr{R}_h^0\mathbf{v} = \mathbf{v} - \sum_{i=j'+1}^{j} \rho_i(\mathbf{x}_i)\varphi_i = \mathbf{v} - \mathscr{R}_h\mathbf{v} + \sum_{i=1}^{j'} \rho_i(\mathbf{x}_i)\varphi_i,$$

where $\rho_i \stackrel{\text{def}}{=} \hat{\rho}_i \circ F_{\Delta_i}^{-1}$, and $\hat{\rho}_i$ is obtained in (3.1). Owing to Lemma 3.2, it remains to estimate $\rho_i(\mathbf{x}_i)\varphi_i$. Notice that for $0 \leqslant m \leqslant k$,

$$\|\varphi_i\|_{m,K} \leqslant Ch^{1-m}.$$

By Lemma 2.2, we have

$$|\rho_i(\mathbf{x}_i)| \leqslant \|\rho_i\|_{0,\infty,K^i} \leqslant C\|\hat{\rho}_i\|_{0,\infty,\hat{K}^i},$$

where K^i is a 2-simplex containing $\mathbf{x}_i$. By the Sobolev inequality, we have

$$|\rho_i(\mathbf{x}_i)| \leqslant C\|\hat{\rho}_i\|_{0,\infty,\hat{K}^i} \leqslant C\|\hat{\rho}_i\|_{2,\hat{K}^i}.$$

Notice that the norms $\|\cdot\|_{2,\hat{K}}$ and $\|\cdot\|_{\hat{K}}$ on $\mathbf{P}_k(\widehat{K}^i)$ do not depend on h, in other words, these norms are equivalent on $\mathbf{P}_k(\widehat{K}^i)$;

$$\|\hat{\rho}_i\|_{2,\hat{K}^i} \leqslant C\|\hat{\rho}_i\|_{\hat{K}^i},$$

where the constant C is independent of h. For each $\hat{\mathbf{v}}$, we have

$$\begin{aligned}\|\hat{\rho}_i\|_{\hat{K}^i} &\leqslant C\|\hat{\rho}_i - \hat{\mathbf{v}}\|_{\hat{K}^i} + C\|\hat{\mathbf{v}}\|_{\hat{K}^i} \leqslant C(\|\rho_i - \mathbf{v}\|_{K^i} + \|\mathbf{v}\|_{K^i})|\det B_{K^i}|^{-1/2} \\ &\leqslant Ch^{-1}(\|\rho_i - \mathbf{v}\|_{K^i} + \|\mathbf{v}\|_{K^i}).\end{aligned}$$

By Lemma 3.1, one has $\|\rho_i - \mathbf{v}\|_{K^i} \leqslant Ch\|\mathbf{v}\|_{1,\Delta_i}$. Hence, we have

$$|\rho_i(\mathbf{x}_i)| \leqslant C(\|\mathbf{v}\|_{1,\Delta_i} + h^{-1}\|\mathbf{v}\|_{K^i}).$$

Applying Lemma 3.3 we have

$$\|\mathbf{v}\|_{K^i} \leqslant Ch\|\nabla\mathbf{v}\|_{\tilde{\Delta}_i},$$

where $\widetilde{\Delta}_i$ is a subset of $\widetilde{\Delta}_K$ corresponding to $\Delta_i \supset K_i$. Therefore, we have

$$\|\mathbf{v} - \mathscr{R}_h^0\mathbf{v}\|_{m,K} \leqslant Ch^{1-m}\|\mathbf{v}\|_{1,\Delta_i} + Ch^{1-m}\|\nabla\mathbf{v}\|_{\tilde{\Delta}_i}.$$

We now consider $\mathscr{I}_h^0$. Let $\mathbf{v} \in W_0^{2,s}(\Omega)^2$. Since

$$\mathbf{v} - \mathscr{I}_h^0\mathbf{v} = \mathbf{v} - \mathscr{I}_h\mathbf{v} + \sum_{i=1}^{j'}\mathbf{v}(\mathbf{x}_i)\varphi_i,$$

we need to estimate $\mathbf{v}(\mathbf{x}_i)\varphi_i$. Consider $\mathbf{z} = (z_1, z_2)$ in Fig. 2. Then, for $(z_1, z_0) \in \gamma$ we have $\mathbf{v}(\mathbf{z}) = \mathbf{v}(z_1, z_0) + \partial_x\mathbf{v}(\tilde{\mathbf{z}})(z_1 - z_0) = \partial_x\mathbf{v}(\tilde{\mathbf{z}})(z_1 - z_0)$, where $\tilde{\mathbf{z}}$ is a point between (z_1, z_0) and $\mathbf{z}$. Since $s > 2$, we have $W^{2,s}(\Omega)^2 \subset \mathscr{C}^1$ by Sobolev embedding theorem. Now that $|z_1 - z_0| \leqslant \delta(\Omega, \Omega_h) \leqslant Ch^2$, we have that for $\mathbf{x}_i \in \partial\Omega_h$, $|\mathbf{v}(\mathbf{x}_i)| \leqslant Ch^2\|\nabla\mathbf{v}\|_{0,\infty,\tilde{K}} \leqslant Ch^2\|\mathbf{v}\|_{2,s,\tilde{K}}$, which completes the proof. □

4. Construction of finite element spaces satisfying LBB conditions

In this section, we construct a finite element space which satisfies the inf-sup condition. On the Lagrange finite elements we consider polynomials as in [3]. Let K be an arbitrary triangle in $\mathscr{T}_h$ with vertices $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Denote by f_i the side opposite to $\mathbf{a}_i$, and by $\mathbf{n}_i$ and $\mathbf{t}_i$ the unit outward normal and unit tangent vectors to f_i like in Fig. 3.

Let

$$\rho_1 \overset{\text{def}}{=} \mathbf{n}_1\lambda_2\lambda_3, \quad \rho_2 \overset{\text{def}}{=} \mathbf{n}_2\lambda_1\lambda_3, \quad \rho_3 \overset{\text{def}}{=} \mathbf{n}_3\lambda_1\lambda_2.$$

Consider the polynomial subspace, denoted by $\mathscr{P}_1$, of $\mathbf{P}_2(K)$:

$$\mathscr{P}_1(K) \overset{\text{def}}{=} \mathbf{P}_1(K) \oplus \text{span}\{\rho_1, \rho_2, \rho_3\}.$$

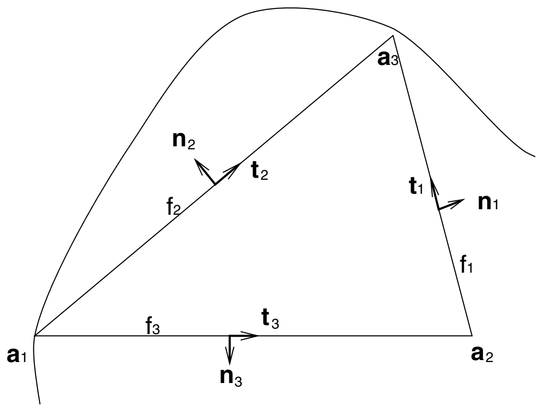

Fig. 3. Triangle in $\mathscr{T}_h$.

Define finite element spaces

$$X_h \stackrel{\text{def}}{=} \{\mathbf{v} \in \mathscr{C}^0(\Omega_h) : \mathbf{v}|_K \in \mathscr{P}_1(K) \quad \text{for all } K \in \mathscr{T}_h\} \cap H_0^1(\Omega_h)^2,$$
$$Q_h \stackrel{\text{def}}{=} \{q \in L_a^2(\Omega_h) : q|_K \in P_0(K) \quad \text{for all } K \in \mathscr{T}_h\}.$$

It is clear that any polynomial $\rho \in \mathscr{P}_1(K)$ is uniquely determined by

$$\rho(\mathbf{a}_i), \quad \text{and} \quad \int_{f_i} \rho \cdot \mathbf{n}_i \, d\sigma, \quad 1 \leqslant i \leqslant 3,$$

where $\mathbf{a}_i$ are the three vertices of K, and f_i are the three sides of K. Any polynomial $\rho_K \in \mathscr{P}_1(K)$ has the form:

$$\rho_K = \sum_{i=1}^{3} \rho(\mathbf{a}_i)\lambda_i + \sum_{i=1}^{3} \alpha_i \rho_i, \tag{4.1}$$

with $\alpha_i \in \mathbb{R}$. In other words, any polynomial $\rho_K \in \mathscr{P}_1(K)$ is of the form

$$\rho_K(\mathbf{x}) = \mathbf{c}_1\lambda_1(\mathbf{x}) + \mathbf{c}_2\lambda_2(\mathbf{x}) + \mathbf{c}_3\lambda_3(\mathbf{x}) + \alpha_1\rho_1 + \alpha_2\rho_2 + \alpha_3\rho_3, \tag{4.2}$$

where $\mathbf{c}_i \in \mathbb{R}^2$ and $\alpha_i \in \mathbb{R}$ are constants. We use the L^2-interpolation $\mathscr{R}_h^0\mathbf{v}|_K \in \mathbf{P}_k(K)$ where $k = 1$. We now define the operator $\Pi_h \in \mathscr{L}(X; X_h)$. In order to define $\Pi_h\mathbf{v}$ for each $\mathbf{v} \in X$, we need to find the constants $\mathbf{c}_i$ and α_i in (4.2). The number of unknowns on each K is 9.

We define Π_h in several ways depending on a 2-simplex K in $\mathscr{T}_h$. Let $\mathbf{v} \in X$ be given. Then the first condition for the definition of $\Pi_h\mathbf{v}$ is given by the following:

(Π_0) Let K be a 2-simplex in $\mathscr{T}_h$. The $\Pi_h\mathbf{v}$ satisfies

$$\Pi_h\mathbf{v}(\mathbf{a}) = \mathscr{R}_h^0\mathbf{v}(\mathbf{a}) \quad \text{for each vertex } \mathbf{a} \text{ of } K. \tag{4.3}$$

If a vertex $\mathbf{a}$ is on $\partial\Omega_h$, then $\Pi_h\mathbf{v}(\mathbf{a}) = 0$.

The second condition for $\Pi_h\mathbf{v}$ is given by the following:

(Π_1) If $K \in \mathscr{T}_h$ is interior of Ω_h, that is, K has no vertex sharing with $\partial\Omega_h$, then the second condition of $\Pi_h\mathbf{v}$ is given by

$$\int_f (\Pi_h\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}\,\mathrm{d}\sigma = 0, \tag{4.4}$$

for each side f of K, where $\mathbf{n}$ is the unit outward normal vector on each side f.

In this case, the side f corresponds to f_5, f_{18}, f_{19} for $K_9, f_{19}, f_{20}, f_{21}$ for K_{10}, and so on. The number of unknown variables is 9, and the number of equations is also 9. In [6], Π_h is defined in this way.

(Π_2) In case that $K \in \mathscr{T}_h$ has *two* vertices $\mathbf{a}^1$, $\mathbf{a}^3$ in $\partial\Omega_h$ like K_1, K_3, K_4, K_7 and K_8 in Fig. 4. For all sides f^1, f^3 of K not on $\partial\Omega_h$, the second condition of $\Pi_h\mathbf{v}$ is given by

$$\int_{f^1\cup f^3} \Pi_h\mathbf{v} \cdot \mathbf{n}\,\mathrm{d}\sigma = \int_{\partial K} \mathbf{v} \cdot \mathbf{n}\,\mathrm{d}\sigma, \tag{4.5}$$

where $\mathbf{n}$ is the unit outward normal vector on each face f^i.

In this case, since we want to find $\Pi_h\mathbf{v}$ in $X_h \subset H_0^1(\Omega)^2$, $\Pi_h\mathbf{v}$ is zero on the side f^2 having the ends $\mathbf{a}^1$, $\mathbf{a}^3$ like f_2 of K_1, f_6 of K_3, and so on. Hence, we have from (4.1) and (4.2) that $\mathbf{c}_1 = \mathbf{c}_3 = 0$. Furthermore, $\alpha_2 = 0$ since $\Pi_h\mathbf{v}$ is zero on f^2. Therefore, $\mathbf{c}_2$ and α_1, α_3 are undetermined yet, in other words, the number of unknowns is 4. From (4.3) $\mathbf{c}_2$ is determined, therefore, owing to (4.5) we need one more condition, which will be specified in (Π_4) and (Π_5). (Notice that f^i and f_i are different face notations.)

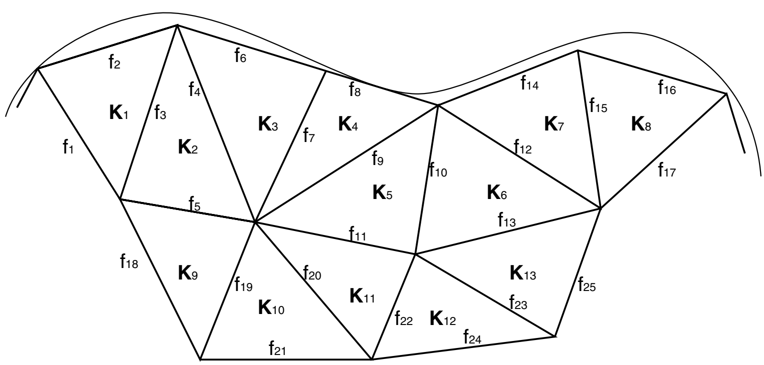

Fig. 4. Triangulations.

(Π_3) In case that $K \in \mathcal{T}_h$ has *one* vertex $\mathbf{a}^2$ in $\partial\Omega_h$ like K_2, K_5 and K_6 in Fig. 4. For the side f^2 of K not having $\mathbf{a}^2$ as its end point, the second condition of $\Pi_h\mathbf{v}$ is given by

$$\int_{f^2} (\Pi_h\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}\,\mathrm{d}\sigma = 0. \tag{4.6}$$

For the other two sides f^1 and f^3 of K,

$$\int_{f^1\cup f^3} (\Pi_h\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}\,\mathrm{d}\sigma = 0. \tag{4.7}$$

In this case, f^2 corresponds to f_5 of K_2, f_{11} of K_5 and f_{13} of K_6. The other two sides f^3, f^1 of K correspond to f_3, f_4 of K_2, f_9, f_{10} of K_5, and so on. From (4.3), $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ are determined, and owing to (4.6) and (4.7), like the case (Π_2) one more condition is needed.

To the cases (Π_2) and (Π_3) we add the following third condition:

(Π_4) Let K^1 and K^2 be two 2-simplices in (Π_2) or (Π_3) sharing a 1-face f with. Then, on f the third condition of $\Pi_h\mathbf{v}$ is given by

$$\int_{f} (\Pi_h\mathbf{v}|_{K^1} - \Pi_h\mathbf{v}|_{K^2}) \cdot \mathbf{n}\,\mathrm{d}\sigma = 0, \tag{4.8}$$

where $\mathbf{n}$ is the unit outward normal vector of K^1 on f. Here, f corresponds to f_3 of K_1 and K_2, f_4 of K_2 and K_3, f_7 of K_3 and K_4, f_{15} of K_7 and K_8, and so on. If the number of 2-simplices in cases (Π_2) or (Π_3) is s, then from (Π_4) we have one degree of freedom left. Finally, we add the last condition to (Π_4).

(Π_5) Let K be a fixed 2-simplex in (Π_3), and let f be one of the face of K with one end on $\partial\Omega_h$. Then, on f the forth condition of $\Pi_h\mathbf{v}$ is given by

$$\int_{f} (\Pi_h\mathbf{v} - \mathbf{v}) \cdot \mathbf{n}\,\mathrm{d}\sigma = 0. \tag{4.9}$$

Remark. (1) In (Π_5) we take only one 2-simplex K and also only one 1-face of K. If we consider another K' satisfying the condition in (Π_5), then the system to find $\Pi_h\mathbf{v}$ is over determined. (2) With the view of (4.1) and from the definition of Π_h, we have that for each $\mathbf{x} \in K$,

$$\Pi_h\mathbf{v}(\mathbf{x}) = \sum_{i=1}^{3} \mathcal{R}_h^0(\mathbf{a}_i)\lambda_i(\mathbf{x}) + \sum_{i=1}^{3} \alpha_i\rho_i(\mathbf{x}),$$

where $\mathbf{a}_i$ are the three vertices of K. With the same point of view, the continuity of $\Pi_h\mathbf{v}$ is obtained easily. If $k = 1$ in the definition of $\mathcal{R}_h^0$ in Section 3, then

$$\mathscr{R}_h^0(\mathbf{v})(\mathbf{x}) = \sum_{i=1}^{3} \mathscr{R}_h^0(\mathbf{a}_i)\lambda_i(\mathbf{x}), \tag{4.10}$$

of which error estimate is given in Theorem 3.4.

We finally have a unique $\Pi_h\mathbf{v} \in X_h$ for each $\mathbf{v} \in X$. Let us now estimate the error of $\Pi_h\mathbf{v}$ for each $\mathbf{v} \in X$.

Lemma 4.1. *Assume that the triangulation $\mathscr{T}_h$ is regular. Let $\Omega_0 \subset \Omega_h$ be the union of 2-simplices $K \in \mathscr{T}_h$ having no vertices sharing with $\partial\Omega_h$ like in case (Π_1). Then we have*

$$|\mathbf{v} - \Pi_h\mathbf{v}|_{m,\Omega_0} \leqslant Ch^{1-m}\|\mathbf{v}\|_{1,\Omega_0} \text{ for all } \mathbf{v} \in H_0^1(\Omega)^2 \tag{4.11}$$

for $m = 0$ or 1, with a constant C independent of h and $\mathbf{v}$.

Furthermore, if $s > 2$, then we have that for $0 \leqslant m \leqslant 2$,

$$|\mathbf{v} - \Pi_h\mathbf{v}|_{m,\Omega_0} \leqslant Ch^{2-m}\|\mathbf{v}\|_{2,\Omega_0} \text{ for all } \mathbf{v} \in W_0^{2,s}(\Omega)^2. \tag{4.12}$$

Proof. Let K be a 2-simplex in case (Π_1), for example, K_{10} in Fig. 4. Set $\mathbf{a}_i$, $i = 1, 2, 3$, be the vertices of K, and set f^i be the 1-face of K opposite to $\mathbf{a}_i$. With the view of (4.1) and from the definition (4.3) and (4.4) of Π_h, we have that for each $\mathbf{x} \in K$,

$$\Pi_h\mathbf{v}(\mathbf{x}) = \sum_{j=1}^{3} \mathscr{R}_h^0(\mathbf{a}_j)\lambda_j(\mathbf{x}) + \sum_{j=1}^{3} \alpha_j\rho_j(\mathbf{x}),$$

where

$$\alpha_j = \frac{\int_{f^j}(\mathbf{v} - \mathscr{R}_h^0(\mathbf{a}_j)\lambda_j) \cdot \mathbf{n}_j \, \mathrm{d}\sigma}{\int_{f^j}\rho_j \cdot \mathbf{n}_j \, \mathrm{d}\sigma}. \tag{4.13}$$

Here, $\int_{f^j}\rho_j \cdot \mathbf{n}_j \, \mathrm{d}\sigma = \int_{f^j}\lambda_{j'}\lambda_{j''} \, \mathrm{d}\sigma$ where if $j = 3$ then $j' = 1$ and $j'' = 2$, if $j = 2$ then $j' = 1$ and $j'' = 3$, and if $j = 1$ then $j' = 2$ and $j'' = 3$. We use this index notation for j, j', j'' for short.

From Lemma 2.2, we infer that

$$|\lambda_{j'}\lambda_{j''}|_{m,K} \leqslant C|\det B_K|^{1/2}\|B_K^{-1}\|^m|\hat{\lambda}_{j'}\hat{\lambda}_{j''}|_{m,\hat{K}} \leqslant C|\det B_K|^{1/2}\|B_K^{-1}\|^m \tag{4.14}$$

since $|\hat{\lambda}_{j'}\hat{\lambda}_{j''}|_{m,\hat{K}}$ is a constant independent of h and K. Notice that

$$\int_{f^j}\lambda_{j'}\lambda_{j''} \, \mathrm{d}\sigma = \frac{\mathrm{meas}(f^j)}{\mathrm{meas}(\hat{f}^j)}\int_{\hat{f}^j}\hat{\lambda}_{j'}\hat{\lambda}_{j''} \, \mathrm{d}\hat{\sigma} \tag{4.15}$$

and that

$$\left| \int_{f^j} (\mathbf{v} - \mathscr{R}_h^0 \mathbf{v}) \cdot \mathbf{n}_j \, d\sigma \right| \leqslant \frac{\operatorname{meas}(f^j)}{\operatorname{meas}(\hat{f}^j)} \left[\int_{\hat{f}^j} |\hat{\mathbf{v}} - \widehat{\mathscr{R}_h^0 \mathbf{v}}|^2 \, d\hat{\sigma} \right]^{1/2}.$$

By the trace theorem (Theorem 5.22 in Adams 2.2 or Theorem I.1.5 in [6]),

$$\left[\int_{\hat{f}^j} |\hat{\mathbf{v}} - \widehat{\mathscr{R}_h^0 \mathbf{v}}|^2 \, d\hat{\sigma} \right]^{1/2} \leqslant C \| \hat{\mathbf{v}} - \widehat{\mathscr{R}_h^0 \mathbf{v}} \|_{1,\hat{K}}. \tag{4.16}$$

From Lemma 2.2 and Theorem 3.4, we have

$$\begin{aligned} \| \hat{\mathbf{v}} - \widehat{\mathscr{R}_h^0 \mathbf{v}} \|_{1,\hat{K}} &\leqslant C |\det B_K|^{-1/2} \Big(\| \mathbf{v} - \mathscr{R}_h^0 \mathbf{v} \|_K + \|B_K\| \, |\mathbf{v} - \mathscr{R}_h^0 \mathbf{v}|_{1,K} \Big) \\ &\leqslant C |\det B_K|^{-1/2} h \|\mathbf{v}\|_{1,\Delta_K}, \end{aligned} \tag{4.17}$$

where Δ_K is defined in Lemma 3.2. From (4.13), (4.15), (4.16) and (4.17), we have

$$|\alpha_i| \leqslant C |\det B_K|^{-1/2} h \|\mathbf{v}\|_{1,\Delta_K}.$$

Therefore, from the above and (4.14) we have

$$\left| \sum_i \alpha_i \rho_i \right|_{m,K} \leqslant C h^{1-m} \|\mathbf{v}\|_{1,\Delta_K}, \tag{4.18}$$

where Δ_k is defined in Lemma 3.2. By (4.18) and Theorem 3.4, we have (4.11).

For $\mathbf{v} \in W^{2,s}(\Omega)^2$, $\mathscr{R}_h^0 \mathbf{v} \overset{\text{def}}{=} \mathscr{I}_h^0 \mathbf{v}$ as mentioned before. Instead of (4.17), we have

$$\| \hat{\mathbf{v}} - \widehat{\mathscr{R}_h^0 \mathbf{v}} \|_{1,\hat{K}} \leqslant C |\det B_K|^{-1/2} h^2 \|\mathbf{v}\|_{2,\Delta_K},$$

hence, instead of (4.18), we have

$$\left| \sum_i \alpha_i \rho_i \right|_{m,K} \leqslant C h^{2-m} \|\mathbf{v}\|_{2,\Delta_K},$$

therefore, by combining this with Theorem 3.4 we complete the proof of (4.12). □

Theorem 4.2. *If the triangulation $\mathscr{T}_h$ is regular, and $\delta(\Omega, \Omega_h) \leqslant Ch^2$ holds, then*

$$|\mathbf{v} - \Pi_h \mathbf{v}|_{m,\Omega_h} \leqslant C h^{1-m} \|\mathbf{v}\|_{1,\Omega} \quad \text{for all } \mathbf{v} \in H_0^1(\Omega)^2 \tag{4.19}$$

for $m = 0$ or 1, with a constant C independent of h and $\mathbf{v}$. We also have

$$\int_{\Omega_h} \operatorname{div}(\mathbf{v} - \Pi_h \mathbf{v}) q_h \, d\mathbf{x} = 0 \quad \text{for all } q_h \in Q_h. \tag{4.20}$$

Moreover, if $s > 2$, *then we have that for* $0 \leqslant m \leqslant 2$,

$$|\mathbf{v} - \Pi_h \mathbf{v}|_{m,\Omega_h} \leqslant Ch^{2-m} \|\mathbf{v}\|_{2,\Omega} \quad \textit{for all } \mathbf{v} \in W_0^{2,s}(\Omega)^2 \tag{4.21}$$

Proof. The identity (4.20) is obtained directly from the conditions (4.4), (4.5), (4.6) and (4.7) depending on K by applying the divergence theorem.

In case (Π_1), the proof is given in Lemma 4.1. Hence, it is enough to consider simplices only in the cases (Π_2) and (Π_3). For simplicity we use the labels like in Fig. 5 and in Fig. 6, which Fig. 5 is in case (Π_2) and Fig. 6 in case (Π_3).

Notice that there is finite many such simplices since the triangulation $\mathscr{T}_h$ is regular, and that h times the number of such simplices is a constant depending on Ω, but independent of h. We let $K^i, i = 1, \dots, s$ be such consecutive 2-simplices sharing at least one vertex with $\partial\Omega_h$, and $K^{s+1} = K^1$ and $K^{-1} = K^s$. From the above statement, we know sh is constant. We denote by f^i the face $K^i \cap K^{i+1}$ for $i = 1, \dots, s$, and by $f^{i-1/2}$ the face of K^i between f^i and f^{i-1} for each i. We denote by $\lambda_{i,j}$ the barycentric coordinates λ_j, and by $\mathbf{a}_{i,j}$ the vertices $\mathbf{a}_j$ on K^i. In Fig. 4, K^i corresponds to $K_1, \dots, K_8$ and more, and f^s to f_1 and f^i to $f_3, f_4, f_7, f_9, f_{10}, f_{12}, f_{15}, f_{17}$, and others which do not show up in this picture. In Fig. 4, $f^{i-1/2}$ corresponds to f_2 of K_1, f_6 of K_3, f_5 of K_2, f_{11} of K_5, etc.

Step 1: First we consider K and the face f taken in (Π_5). Then there is an index i such that $K = K^i$ and $1 \leqslant i \leqslant s$, then f is one of f^i and f^{i-1}. Let $f = f^i$. In Fig. 4, for example, K could be K_5 and f be f_{10}. The conditions (4.6) on

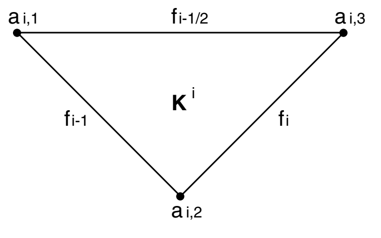

Fig. 5. Case (Π_2).

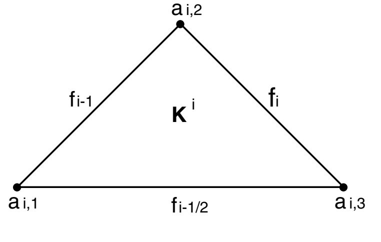

Fig. 6. Case (Π_3).

$f^{i-1/2}$, (4.7) on $f^i \cup f^{i-1}$, and (4.9) on f^i become (4.4) on $f^i, f^{i-1/2}$ and f^{i-1}. Thus we can prove (4.19) on K like in Lemma 4.1. Consider first the case that K^{i+1} is in case (Π_3) like K_6 in Fig. 4. Since we have (4.9) on f^i and (4.8) on the intersection f^i of K^i and K^{i+1}, we have the same situation like K^i.

Step 2: We now consider that K^{i+1} is in case (Π_2), for example, $K^i = K_6$ and $K^{i+1} = K_7$ in Fig. 4. Let $\mathbf{a}_1$ and $\mathbf{a}_3$ be the two vertices of K^{i+1} on $\partial\Omega_h$, and let $\mathbf{a}_2$ the rest vertex not on $\partial\Omega_h$. Owing to (4.8) and (4.9) on f^i, setting $f^{i+1/2}$ be the face of K on $\partial\Omega_h$, we have from (4.5) that

$$\int_{f^{i+1}} \Pi_h \mathbf{v}\,\mathrm{d}\sigma = \int_{f^{i+1/2}\cup f^{i+1}} \mathbf{v}\,\mathrm{d}\sigma. \tag{4.22}$$

Since $\mathscr{R}_h^0(\mathbf{a}_{i+1,1}) = \mathscr{R}_h^0(\mathbf{a}_{i+1,3}) = 0$, and $\Pi_h\mathbf{v} = 0$ on $f^{i+1/2}$, we have

$$\Pi_h\mathbf{v}(\mathbf{x}) = \mathscr{R}_h^0(\mathbf{a}_{i+1,2})\lambda_{i+1,2}(\mathbf{x}) + \alpha_{i+1,1}\rho_{i+1,1}(\mathbf{x}) + \alpha_{i+1,3}\rho_{i+1,3}(\mathbf{x}). \tag{4.23}$$

By (4.8) of Π_h, we have

$$0 = \int_{f^i} (\mathscr{R}_h^0(\mathbf{a}_{i+1,2})\lambda_{i+1,2} - \mathbf{v})\cdot\mathbf{n}_{i+1,3}\,\mathrm{d}\sigma + \alpha_{i+1,3}\int_{f^i}\lambda_{i+1,1}\lambda_{i+1,2}\,\mathrm{d}\sigma.$$

Thus we have

$$\alpha_{i+1,3} = \frac{\int_{f^i}(\mathbf{v} - \mathscr{R}_h^0(\mathbf{a}_{i+1,2})\lambda_{i+1,2})\cdot\mathbf{n}_{i+1,3}\,\mathrm{d}\sigma}{\int_{f^i}\lambda_{i+1,1}\lambda_{i+1,2}\,\mathrm{d}\sigma}.$$

In a similar way in the proof of Lemma 4.1, we obtain

$$|\alpha_{i+1,3}\rho_3|_{m,K^{i+1}} \leqslant Ch^{1-m}\|\mathbf{v}\|_{1,\tilde{\Delta}_{K^{i+1}}}, \tag{4.24}$$

where $\widetilde{\Delta}_{K^{i+1}}$ is defined similarly as in Theorem 3.4. Owing to (4.23), we have that

$$\begin{aligned}\int_{f^{i+1}}(\Pi_h\mathbf{v}-\mathbf{v})\cdot\mathbf{n}_{i+1,1}\,\mathrm{d}\sigma &= \int_{f^{i+1}}(\mathscr{R}_h^0\mathbf{v}(\mathbf{a}_{i+1,2})\lambda_{i+1,2}-\mathbf{v})\cdot\mathbf{n}_{i+1,1}\,\mathrm{d}\sigma \\ &\quad + \alpha_{i+1,1}\int_{f^{i+1}}\lambda_{i+1,2}\lambda_{i+1,3}\,\mathrm{d}\sigma, \\ \int_{f^i}(\Pi_h\mathbf{v}-\mathbf{v})\cdot\mathbf{n}_{i+1,3}\,\mathrm{d}\sigma &= \int_{f^i}(\mathscr{R}_h^0\mathbf{v}(\mathbf{a}_{i+1,2})\lambda_{i+1,2}-\mathbf{v})\cdot\mathbf{n}_{i+1,3}\,\mathrm{d}\sigma \\ &\quad + \alpha_{i+1,3}\int_{f^i}\lambda_{i+1,1}\lambda_{i+1,2}\,\mathrm{d}\sigma.\end{aligned}$$

From (4.5) we have that

$$\int_{f^{i+1/2}}\mathbf{v}\cdot\mathbf{n}_{i+1,2}\,\mathrm{d}\sigma = \int_{f^{i+1}}(\Pi_h\mathbf{v}-\mathbf{v})\cdot\mathbf{n}_{i+1,1}\,\mathrm{d}\sigma + \int_{f^i}(\Pi_h\mathbf{v}-\mathbf{v})\cdot\mathbf{n}_{i+1,3}\,\mathrm{d}\sigma.$$

Hence, we have

$$\begin{aligned}\int_{f^{i+1/2}} \mathbf{v}\cdot\mathbf{n}_{i+1,2}\,\mathrm{d}\sigma &= \int_{f^{i+1}} (\mathscr{R}_h^0\mathbf{v}(\mathbf{a}_{i+1,2})\lambda_{i+1,2}-\mathbf{v})\cdot\mathbf{n}_{i+1,1}\,\mathrm{d}\sigma + \alpha_{i+1,1}\int_{f^{i+1}}\lambda_{i+1,2}\lambda_{i+1,3}\,\mathrm{d}\sigma \\ &\quad + \int_{f^{i}} (\mathscr{R}_h^0\mathbf{v}(\mathbf{a}_{i+1,2})\lambda_{i+1,2}-\mathbf{v})\cdot\mathbf{n}_{i+1,3}\,\mathrm{d}\sigma + \alpha_{i+1,3}\int_{f^{i}}\lambda_{i+1,1}\lambda_{i+1,2}\,\mathrm{d}\sigma.\end{aligned}$$

Thus, we have

$$\alpha_{i+1,1} = \frac{\int_{\partial K^{i+1}}(\mathbf{v}-\mathscr{R}_h^0\mathbf{v}(\mathbf{a}_{i+1,2})\lambda_{i+1,2})\cdot\mathbf{n}\,\mathrm{d}\sigma}{\int_{f^{i+1}}\lambda_{i+1,2}\lambda_{i+1,3}\,\mathrm{d}\sigma} - \alpha_{i+1,3}\frac{\int_{f^{i}}\lambda_{i+1,1}\lambda_{i+1,2}\,\mathrm{d}\sigma}{\int_{f^{i+1}}\lambda_{i+1,2}\lambda_{i+1,3}\,\mathrm{d}\sigma}.$$

Notice that

$$\left|\int_{\partial K^{i+1}}(\mathbf{v}-\mathscr{R}_h^0\mathbf{v}(\mathbf{a}_{i+1,2})\lambda_{i+1,2})\cdot\mathbf{n}\,\mathrm{d}\sigma\right| \leqslant \frac{\operatorname{meas}(\partial K^{i+1})}{\operatorname{meas}(\hat\partial K^{i+1})}\left[\int_{\partial\hat K^{i+1}}|\hat{\mathbf{v}}-\widehat{\mathscr{R}_h^0\mathbf{v}}|^2\,\mathrm{d}\hat\sigma\right]^{1/2}.$$

By the trace theorem in [1], we have

$$\left[\int_{\partial\hat K^{i+1}}|\hat{\mathbf{v}}-\widehat{\mathscr{R}_h^0\mathbf{v}}|^2\,\mathrm{d}\hat\sigma\right]^{1/2} \leqslant C\|\hat{\mathbf{v}}-\widehat{\mathscr{R}_h^0\mathbf{v}}\|_{1,\hat K^{i+1}}.$$

From Lemma 2.2 and Theorem 3.4, we have

$$\begin{aligned}\|\hat{\mathbf{v}}-\widehat{\mathscr{R}_h^0\mathbf{v}}\|_{1,\hat K^{i+1}} &\leqslant C|\det B_K|^{-1/2}\Big(\|\mathbf{v}-\mathscr{R}_h^0\mathbf{v}\|_{K^{i+1}} + \|B_K\|\;|\mathbf{v}-\mathscr{R}_h^0\mathbf{v}|_{1,K^{i+1}}\Big)\\ &\leqslant C|\det B_K|^{-1/2}h^1\|\mathbf{v}\|_{1,\tilde\Delta_{K^{i+1}}}.\end{aligned}$$

Thus, in a similar way in the proof of Lemma 4.1, we have

$$|\alpha_{i+1,1}\rho_{i+1,1}|_{m,K^{i+1}} \leqslant Ch^{1-m}\|\mathbf{v}\|_{1,\tilde\Delta_{K^{i+1}}}. \tag{4.25}$$

Therefore, from (4.10), (4.23)–(4.25), and Theorem 3.4 we have

$$|\mathbf{v}-\Pi_h\mathbf{v}|_{m,K^{i+1}} \leqslant Ch^{1-m}\|\mathbf{v}\|_{1,\tilde\Delta_{K^{i+1}}}, \tag{4.26}$$

where $\Delta_{K^{i+1}}$ is defined similarly as in Theorem 3.4.

Step 3: We also have two cases such that the next 2-simplex K^{i+2} is in case (Π_2) or in case (Π_3). In case (Π_3), we have

$$\Pi_h\mathbf{v} = \mathscr{R}_h^0(\mathbf{a}_{i+2,1})\lambda_{i+2,1} + \mathscr{R}_h^0(\mathbf{a}_{i+2,3})\lambda_{i+2,3} + \sum_{j=1}^{3}\alpha_{i+2,j}\rho_{i+2,j}, \tag{4.27}$$

and in case (Π_2),

$$\Pi_h\mathbf{v} = \mathscr{R}_h^0(\mathbf{a}_{i+2,2})\lambda_{i+2,2} + \alpha_{i+2,1}\rho_{i+2,1} + \alpha_{i+2,3}\rho_{i+2,3}, \tag{4.28}$$

First consider K^{i+2} in case (Π_3). From (4.27) we have that for $j = 1, 2, 3$,

$$(\Pi_h \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{i+2,j} = \mathscr{R}_h^0(\mathbf{a}_{i+2,1})\lambda_{i+2,1} \cdot \mathbf{n}_{i+2,j} - \mathbf{v} \cdot \mathbf{n}_{i+2,j} + \mathscr{R}_h^0(\mathbf{a}_{i+2,3})\lambda_{i+2,3} \cdot \mathbf{n}_{i+2,j} + \alpha_{i+2,j}\lambda_{i+2,j'}\lambda_{i+2,j''}, \qquad (4.29)$$

where $j', j'' \in \{1, 2, 3\}$, and j, j', j'' are all different. Owing to (4.6) and (4.7), we have from (4.27) that

$$\begin{aligned}
&\int_{f^{i+1}} \mathbf{v} \cdot \mathbf{n}_{i+2,3}\,\mathrm{d}\sigma + \int_{f^{i+2}} \mathbf{v} \cdot \mathbf{n}_{i+2,1}\,\mathrm{d}\sigma \\
&\quad = \int_{f^{i+1}} \mathscr{R}_h^0(\mathbf{a}_{i+2,1})\lambda_{i+2,1} \cdot \mathbf{n}_{i+2,3}\,\mathrm{d}\sigma + \int_{f^{i+2}} \mathscr{R}_h^0(\mathbf{a}_{i+2,1})\lambda_{i+2,1} \cdot \mathbf{n}_{i+2,1}\,\mathrm{d}\sigma \\
&\quad + \int_{f^{i+1}} \mathscr{R}_h^0(\mathbf{a}_{i+2,3})\lambda_{i+2,3} \cdot \mathbf{n}_{i+2,3}\,\mathrm{d}\sigma + \int_{f^{i+2}} \mathscr{R}_h^0(\mathbf{a}_{i+2,3})\lambda_{i+2,3} \cdot \mathbf{n}_{i+2,1}\,\mathrm{d}\sigma \\
&\quad + \alpha_{i+2,1} \int_{f^{i+2}} \lambda_{i+2,2}\lambda_{i+2,3}\,\mathrm{d}\sigma + \alpha_{i+2,3} \int_{f^{i+1}} \lambda_{i+2,1}\lambda_{i+2,2}\,\mathrm{d}\sigma.
\end{aligned}$$

Since $\lambda_{i+2,3}(\mathbf{x}) = 0$ for $\mathbf{x} \in f^{i+1}$ and $\lambda_{i+2,1}(\mathbf{x}) = 0$ for $\mathbf{x} \in f^{i+2}$, we have

$$\begin{aligned}
&\int_{f^{i+1}} (\mathbf{v} - \mathscr{R}_h^0(\mathbf{a}_{i+2,1})\lambda_{i+2,1}) \cdot \mathbf{n}_{i+2,3}\,\mathrm{d}\sigma + \int_{f^{i+2}} (\mathbf{v} - \mathscr{R}_h^0(\mathbf{a}_{i+2,3})\lambda_{i+2,3}) \cdot \mathbf{n}_1\,\mathrm{d}\sigma \\
&\quad = \alpha_{i+2,1} \int_{f^{i+2}} \lambda_{i+2,2}\lambda_{i+2,3}\,\mathrm{d}\sigma + \alpha_{i+2,3} \int_{f^{i+1}} \lambda_{i+2,1}\lambda_{i+2,2}\,\mathrm{d}\sigma. \qquad (4.30)
\end{aligned}$$

Since $\mathbf{n}_{i+2,3} = -\mathbf{n}_{i+1,1}$, we have by (4.8) that

$$\int_{f^{i+1}} (\Pi_h \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{i+2,3}\,\mathrm{d}\sigma = -\int_{f^{i+1}} (\Pi_h \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{i+1,1}\,\mathrm{d}\sigma. \qquad (4.31)$$

Thus if K^{i+1} is in case (Π_3) then we have from (4.28) for $i+1$ that

$$\int_{f^{i+1}} (\Pi_h \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{i+1,1}\,\mathrm{d}\sigma = \int_{f^{i+1}} (\mathscr{R}_h^0(\mathbf{a}_{i+1,3})\lambda_{i+1,3} - \mathbf{v}) \cdot \mathbf{n}_{i+1,1}\,\mathrm{d}\sigma + \alpha_{i+1,1} \int_{f^{i+1}} \lambda_{i+1,2}\lambda_{i+1,3}\,\mathrm{d}\sigma. \qquad (4.32)$$

From (4.29), (4.31) and (4.32), we have

$$\begin{aligned}
&\int_{f^{i+1}} (\mathscr{R}_h^0(\mathbf{a}_{i+1,3})\lambda_{i+1,3} - \mathbf{v}) \cdot \mathbf{n}_{i+1,1}\,\mathrm{d}\sigma + \alpha_{i+1,1} \int_{f^{i+1}} \lambda_{i+1,2}\lambda_{i+1,3}\,\mathrm{d}\sigma \\
&\quad = -\int_{f^{i+1}} (\mathscr{R}_h^0(\mathbf{a}_{i+2,1})\lambda_{i+2,1} - \mathbf{v}) \cdot \mathbf{n}_{i+2,3}\,\mathrm{d}\sigma - \alpha_{i+2,3} \int_{f^{i+1}} \lambda_{i+2,1}\lambda_{i+2,2}\,\mathrm{d}\sigma.
\end{aligned}$$

Therefore, as we have done in the above, we obtain (4.24) for $i+2$. If K^{i+1} is in case (Π_2) then we have from (4.28) for $i+1$ that

$$\int_{f^{i+1}} (\Pi_h \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{i+1,1}\,\mathrm{d}\sigma = \int_{f^{i+1}} (\mathscr{R}_h^0(\mathbf{a}_{i+1,2})\lambda_{i+1,2} - \mathbf{v}) \cdot \mathbf{n}_{i+1,1}\,\mathrm{d}\sigma + \alpha_{i+1,1}\int_{f^{i+1}} \lambda_{i+1,2}\lambda_{i+1,3}\,\mathrm{d}\sigma.$$

From (4.29) and (4.31), we have

$$\int_{f^{i+1}} (\mathscr{R}_h^0(\mathbf{a}_{i+1,2})\lambda_{i+1,2} - \mathbf{v}) \cdot \mathbf{n}_{i+1,1}\,\mathrm{d}\sigma + \alpha_{i+1,1}\int_{f^{i+1}} \lambda_{i+1,2}\lambda_{i+1,3}\,\mathrm{d}\sigma$$
$$= -\int_{f^{i+1}} (\mathscr{R}_h^0(\mathbf{a}_{i+2,1})\lambda_{i+2,1} - \mathbf{v}) \cdot \mathbf{n}_{i+2,3}\,\mathrm{d}\sigma - \alpha_{i+2,3}\int_{f^{i+1}} \lambda_{i+2,1}\lambda_{i+2,2}\,\mathrm{d}\sigma.$$

Therefore, as we have done in the above, we obtain (4.24) for $i+2$ in this case. Hence if we repeat the same process then from (4.30) we obtain (4.24) for $\alpha_{i+2,1}\rho_{i+2,1}$.

Step 4: Now we assume that K^{i+2} is in case (Π_2). Then from (4.28), we have that for $j = 1, 3$,

$$\Pi_h \mathbf{v} \cdot \mathbf{n}_{i+2,j} = \mathscr{R}_h^0(\mathbf{a}_{i+2,2})\lambda_{i+2,2} \cdot \mathbf{n}_{i+2,j} + \alpha_{i+2,j}\lambda_{i+2,j'}\lambda_{i+2,j''}, \tag{4.33}$$

where $j', j'' \in \{1, 2, 3\}$, and j, j', j'' are all different. Owing to (4.5), we have from (4.28) that

$$\int_{f^{i+1}} \mathbf{v} \cdot \mathbf{n}_3\,\mathrm{d}\sigma + \int_{f^{i+3/2}} \mathbf{v} \cdot \mathbf{n}_2\,\mathrm{d}\sigma + \int_{f^{i+2}} \mathbf{v} \cdot \mathbf{n}_1\,\mathrm{d}\sigma$$
$$= \int_{f^{i+1}} \mathscr{R}_h^0(\mathbf{a}_{i+2,2})\lambda_{i+2,2} \cdot \mathbf{n}_{i+2,3}\,\mathrm{d}\sigma + \int_{f^{i+2}} \mathscr{R}_h^0(\mathbf{a}_{i+2,2})\lambda_{i+2,2} \cdot \mathbf{n}_{i+2,1}\,\mathrm{d}\sigma$$
$$+ \alpha_{i+2,1}\int_{f^{i+2}} \lambda_{i+2,2}\lambda_{i+2,3} + \alpha_{i+2,3}\int_{f^{i+1}} \lambda_{i+2,1}\lambda_{i+2,2}.$$

Consider (4.31) on $f^{i+1} = K^{i+1} \cap K^{i+2}$. If K^{i+1} is in case (Π_3) then we have from (4.32), (4.31) and (4.33) that

$$\int_{f^{i+1}} (\mathscr{R}_h^0(\mathbf{a}_{i+1,3})\lambda_{i+1,3} - \mathbf{v}) \cdot \mathbf{n}_{i+1,1}\,\mathrm{d}\sigma + \alpha_{i+1,1}\int_{f^{i+1}} \lambda_{i+1,2}\lambda_{i+1,3}\,\mathrm{d}\sigma$$
$$= -\int_{f^{i+1}} (\mathscr{R}_h^0(\mathbf{a}_{i+2,2})\lambda_{i+2,2} - \mathbf{v}) \cdot \mathbf{n}_{i+2,3}\,\mathrm{d}\sigma - \alpha_{i+2,3}\int_{f^{i+1}} \lambda_{i+2,1}\lambda_{i+2,2}\,\mathrm{d}\sigma.$$

Thus, we obtain (4.24) for $i+2$, and also (4.26) in a similar way in Step 2. We also obtain (4.26) for K^{i+1} in case (Π_2) in a similar way in Step 3. By summation from $i = 1$, to s we complete the proof of (4.19).

For $\mathbf{v} \in W_0^{2,s}(\Omega)^2$, instead of (4.24) and (4.25) as in the proof of Lemma 4.1, we obtain that the left sides are bounded by $Ch^{2-m}\|\mathbf{v}\|_{2,\tilde{\Delta}_{K^{i+1}}}$. Hence, we obtain

$$|\mathbf{v} - \Pi_h \mathbf{v}|_{m,K^{i+1}} \leqslant Ch^{2-m}\|\mathbf{v}\|_{2,\tilde{\Delta}_{K^{i+1}}}.$$

As above, if we follow each step, then we can complete the proof of (4.21). □

Remark. If we take $m = 1$ in (4.19), then we have

$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_1 \leqslant C\|\mathbf{v}\|_1,$$

which implies (1.4). Thus, we have shown that (1.2) is solvable in $X_h \times Q_h$.

5. Error estimate

We assume that $\partial\Omega$ is of class $\mathscr{C}^2$ and $\mathbf{f} \in L^s(\Omega)^2, 2 < s < \infty$. We denote by $(\mathbf{u}, p)$ the solution of (1.1), then by Proposition 2.1 we have

$$\mathbf{u} \in W^{2,s}(\Omega)^2, \quad p \in W^{1,s}(\Omega).$$

By Sobolev embedding theorem, we have $W^{2,s}(\Omega)^2 \subset \mathscr{C}^1(\Omega)^2$ and $W^{1,s}(\Omega) \subset \mathscr{C}^0(\Omega)$.

Suppose that $\Omega_h \subset \Omega$ and $\delta(\Omega_h, \Omega) \leqslant Ch^2$. By setting zero outside of Ω_h, we extend functions $\mathbf{v}_h \in X_h$ and $q_h \in Q_h$ on Ω, and we may consider the functions defined on Ω. We denote these functions by the same notations if there is no confusion. We denote the L^2-inner products on Ω by $\langle \cdot, \cdot \rangle_\Omega$, and on Ω_h by $\langle \cdot, \cdot \rangle_{\Omega_h}$. Notice that for all $\mathbf{v}_h \in X_h$,

$$\begin{aligned} -\int_{\Omega_h} \Delta\mathbf{u} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} &= -\sum_{K \in \mathscr{T}_h} \int_K \Delta\mathbf{u} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} \\ &= \sum_{K \in \mathscr{T}_h} \int_K \nabla\mathbf{u} \cdot \nabla\mathbf{v}_h \,\mathrm{d}\mathbf{x} - \sum_{K \in \mathscr{T}_h} \int_{\partial K} \partial_n \mathbf{u} \cdot \mathbf{v}_h \,\mathrm{d}\sigma \end{aligned}$$

and that

$$\int_{\Omega_h} \nabla p \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} = -\sum_{K \in \mathscr{T}_h} \int_K p \,\mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{K \in \mathscr{T}_h} \int_{\partial K} p\mathbf{v}_h \cdot \mathbf{n} \,\mathrm{d}\sigma.$$

Since $\mathbf{v}_h \in X_h \subset \mathscr{C}^0$, we have

$$-\int_{\Omega_h} \Delta\mathbf{u} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} = \langle \nabla\mathbf{u}, \nabla\mathbf{v}_h \rangle_{\Omega_h},$$

and

$$\int_{\Omega_h} \nabla p \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} = -\langle p, \mathrm{div}\, \mathbf{v}_h \rangle_{\Omega_h}.$$

Taking the inner product with $\mathbf{v} = \mathbf{v}_h \in X_h$ in (1.1), we have

$$\langle \nabla\mathbf{u}, \nabla\mathbf{v}_h \rangle_{\Omega_h} - \langle p, \mathrm{div}\, \mathbf{v}_h \rangle_{\Omega_h} = \langle \mathbf{f}, \mathbf{v}_h \rangle_{\Omega_h}. \tag{5.1}$$

Subtract (1.2) from (5.1) to have

$$\langle \nabla(\mathbf{u}-\mathbf{u}_h), \nabla \mathbf{v}_h \rangle_{\Omega_h} = \langle p - p_h, \operatorname{div} \mathbf{v}_h \rangle_{\Omega_h} \tag{5.2}$$

for all $\mathbf{v}_h \in X_h$. Define

$$V \stackrel{\text{def}}{=} \{\mathbf{v} \in X : \langle q, \operatorname{div} \mathbf{v} \rangle_{\Omega} = 0 \text{ for all } q \in Q\},$$
$$V_h \stackrel{\text{def}}{=} \{\mathbf{v}_h \in X_h : \langle q_h, \operatorname{div} \mathbf{v}_h \rangle_{\Omega_h} = 0 \text{ for all } q_h \in Q_h\}.$$

For $\mathbf{v}_h \in V_h$ and $q_h \in Q_h$, by (5.2) we have

$$\begin{aligned}
&\langle \nabla(\mathbf{u}-\mathbf{u}_h), \nabla(\mathbf{u}-\mathbf{u}_h) \rangle_{\Omega_h} \\
&\quad = \langle \nabla(\mathbf{u}-\mathbf{u}_h), \nabla(\mathbf{u}-\mathbf{v}_h) \rangle_{\Omega_h} + \langle \nabla(\mathbf{u}-\mathbf{u}_h), \nabla(\mathbf{v}_h-\mathbf{u}_h) \rangle_{\Omega_h} \\
&\quad = \langle \nabla(\mathbf{u}-\mathbf{u}_h), \nabla(\mathbf{u}-\mathbf{v}_h) \rangle_{\Omega_h} + \langle p - p_h, \operatorname{div}(\mathbf{v}_h-\mathbf{u}_h) \rangle_{\Omega_h} \\
&\quad \leqslant \|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{\Omega_h} \|\nabla(\mathbf{u}-\mathbf{v}_h)\|_{\Omega_h} + \|p-q_h\|_{\Omega_h} \|\nabla(\mathbf{v}_h-\mathbf{u}_h)\|_{\Omega_h},
\end{aligned}$$

since for all $q_h \in Q_h$,

$$\langle p - p_h + q_h - q_h, \operatorname{div}(\mathbf{v}_h-\mathbf{u}_h) \rangle_{\Omega_h} = \langle p - q_h, \operatorname{div}(\mathbf{v}_h-\mathbf{u}_h) \rangle_{\Omega_h}.$$

By Young's inequality, we have

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{\Omega_h}^2 \leqslant 3\|\nabla(\mathbf{u}-\mathbf{v}_h)\|_{\Omega_h}^2 + 3\|p-q_h\|_{\Omega_h}^2$$

for all $\mathbf{v}_h \in V_h$ and $q_h \in Q_h$. Thus, we have the following:

Proposition 5.1. *Let* $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times L_a^2(\Omega)$ *be a solution of* (1.1) *and* $(\mathbf{u}_h, p_h) \in X_h \times Q_h$ *a solution of* (1.2). *Then, we have*

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{\Omega_h} \leqslant 3 \inf_{\mathbf{v}_h \in V_h} \|\nabla(\mathbf{u}-\mathbf{v}_h)\|_{\Omega_h} + 3 \inf_{q_h \in Q_h} \|p-q_h\|_{\Omega_h}.$$

Since the usual interpolation $\mathscr{I}_h^0\mathbf{v}$ defined by $\mathscr{I}_h^0\mathbf{v}(\mathbf{x}_i) = \mathbf{v}(\mathbf{x}_i)$ at each node for $\mathbf{v} \in H_0^1(\Omega)^2$ is not be defined, L^2-interpolation $\mathscr{R}_h^0$ is introduced in Section 3. However, we have already known that for $s \geqslant 2$ solution $\mathbf{u}$ of (1.1) belongs to $W^{2,s}(\Omega)^2$, so that $\mathbf{u} \in \mathscr{C}^0(\Omega)^2$ by Sobolev embedding theorem. Thus, for each $\mathbf{v} \in W^{2,s}(\Omega)^2$ with $s > 2$, $\mathscr{I}_h^0\mathbf{v}$ makes sense, and $\mathscr{R}_h^0\mathbf{v} = \mathscr{I}_h^0\mathbf{v}$. We also have the following estimate:

$$\|\mathbf{v} - \mathscr{I}_h^0\mathbf{v}\|_K + h\|\nabla(\mathbf{v} - \mathscr{I}_h^0\mathbf{v})\|_K \leqslant Ch^2\|\nabla^2\mathbf{v}\|_K,$$

for $K \in \mathscr{T}_h$. Consult with the statement following Lemma 3.2 or refer to [4,5,11].

Since $\mathscr{R}_h^0\mathbf{v} = \mathscr{I}_h^0\mathbf{v}$ for $\mathbf{v} \in W^{2,s}(\Omega)^2$, for the construction of the finite element space and Π_h, we use $\mathscr{I}_h$. From Proposition 5.1, we have

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{\Omega_h} \leqslant C\|\nabla(\mathbf{u}-\Pi_h\mathbf{u})\|_{\Omega_h} + C \inf_{q_h \in Q_h} \|p-q_h\|_{\Omega_h}.$$

On the other hand, by taking q_h as the mean value of p on each K for $K \in \mathscr{T}_h$, we have

$$\|p - q_h\|_{\Omega_h} \leqslant h\|\nabla p\|_{\Omega}. \tag{5.3}$$

Therefore, by (4.21) we have

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{\Omega_h} \leqslant Ch\Big(|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega}\Big). \tag{5.4}$$

Hence, by using the projection Π_h, we have the error estimates.

Theorem 5.2. *Assume that Ω is bounded open subset in $\mathbb{R}^2$ of class $\mathscr{C}^2, \mathbf{f} \in L^s(\Omega)^2$ with $2 < s < \infty$, and $\delta(\Omega_h, \Omega) \leqslant Ch^2$. Let $(\mathbf{u}, p)$ be a solution of* (1.1) *given in Proposition* 2.1, *and let $(\mathbf{u}_h, p_h) \in X_h \times Q_h$ be a solution of* (1.2). *Then*,

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{\Omega_h} + \|p - p_h\|_{\Omega_h} \leqslant Ch\Big(|\mathbf{u}|_{2,\Omega} + |p|_{1,\Omega}\Big).$$

Proof. It remain only to estimate the error of the pressure. Look at the pressures p and p_h. We have from (5.2)

$$\langle p_h - q_h, \operatorname{div} \mathbf{v}_h\rangle_{\Omega_h} = \langle \nabla(\mathbf{u}_h - \mathbf{u}), \nabla \mathbf{v}_h\rangle_{\Omega_h} + \langle p - q_h, \operatorname{div} \mathbf{v}_h\rangle_{\Omega_h}$$

for all $\mathbf{v}_h \in X_h$ and $q_h \in Q_h$. By the inf-sup condition (1.3), we have

$$\|p_h - q_h\|_{\Omega_h} \leqslant \frac{1}{\beta} \sup_{\mathbf{v}_h \in X_h} \frac{\langle p_h - q_h, \operatorname{div} \mathbf{v}_h\rangle_{\Omega_h}}{\|\nabla \mathbf{v}_h\|_{\Omega_h}} \leqslant \frac{1}{\beta}\Big(\|\nabla(\mathbf{u}_h - \mathbf{u})\|_{\Omega_h} + \|p - q_h\|_{\Omega_h}\Big)$$

for all $q_h \in Q_h$. Thus,

$$\begin{aligned} \|p - p_h\| &\leqslant \inf_{q_h \in Q_h} \Big(\|p - q_h\|_{\Omega_h} + \|p_h - q_h\|_{\Omega_h}\Big) \\ &\leqslant \inf_{q_h \in Q_h} \left(\left(1 + \frac{1}{\beta}\right)\|p - q_h\|_{\Omega_h} + \frac{1}{\beta}\|\nabla(\mathbf{u}_h - \mathbf{u})\|_{\Omega_h}\right). \end{aligned}$$

By (5.3) and (5.4), we complete the proof. □

Remark. Since $(\mathbf{u}, p) \in W^{2,s}(\Omega)^2 \times \big(W^{1,s}(\Omega) \cap L^2_a(\Omega)\big)$, we have

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{\Omega_h} + \|p - p_h\|_{\Omega_h} \leqslant Ch,$$

where the constant C does not depend on h.

For the L^2 error we use a duality argument.

Theorem 5.3. *With the same assumptions in Theorem* 5.2, *we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\Omega_h} \leqslant Ch^2\Big(\|\mathbf{u}\|_{2,\Omega} + |p|_{1,\Omega}\Big).$$

Proof. We have by the duality argument that

$$\|\mathbf{u}-\mathbf{u}_h\|_{\Omega_h} = \sup_{\mathbf{g}\in L^2(\Omega_h)^2} \frac{\langle \mathbf{g}, \mathbf{u}-\mathbf{u}_h\rangle_{\Omega_h}}{\|\mathbf{g}\|_{\Omega_h}}. \tag{5.5}$$

We extend $\mathbf{g}$ on $\Omega \cup \Omega_h$ by setting zero outside Ω_h. By restricting $\mathbf{g}$ on Ω we have $\mathbf{g} \in L^2(\Omega)^2$. Thus, there is a unique solution for each $\mathbf{g} \in L^2(\Omega)^2$, there is a unique solution $(\mathbf{v}_\mathbf{g}, \mu_\mathbf{g})$ such that

$$\begin{cases} \langle \nabla \mathbf{v}_\mathbf{g}, \nabla \mathbf{f}w\rangle_\Omega + \langle \mu_\mathbf{g}, \operatorname{div} \mathbf{f}w\rangle_\Omega = \langle \mathbf{g}, \mathbf{f}w\rangle_\Omega \text{ for all } \mathbf{f}w \in X_\Omega, \\ \langle \nabla \mathbf{v}_\mathbf{g}, \mu\rangle_\Omega = 0 \text{ for all } \mu \in L^2(\Omega). \end{cases} \tag{5.6}$$

Then, by the regularity of Stokes equations on smooth domain Ω, we have

$$\|\mathbf{v}_\mathbf{g}\|_{2,\Omega} + \|\mu_\mathbf{g}\|_{1,\Omega} \leqslant C\|\mathbf{g}\|_\Omega. \tag{5.7}$$

Taking $\mathbf{f}w = \mathbf{u}-\mathbf{u}_h$ in (5.6), we get

$$\langle \nabla \mathbf{v}_\mathbf{g}, \nabla(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} + \langle \mu_\mathbf{g}, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} = \langle \mathbf{g}, \mathbf{u}-\mathbf{u}_h\rangle_{\Omega_h},$$

By (5.2), we have

$$\langle \nabla \mathbf{v}_h, \nabla(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} = \langle p-p_h, \operatorname{div} \mathbf{v}_h\rangle_{\Omega_h} = \langle p, \operatorname{div} \mathbf{v}_h\rangle_{\Omega_h} = \langle p-q_h, \operatorname{div} \mathbf{v}_h\rangle_{\Omega_h}$$

for all $\mathbf{v}_h \in V_h$ and $q_h \in Q_h$. Since $\mathbf{v}_\mathbf{g} \in V$, we have

$$\langle \operatorname{div} \mathbf{v}_\mathbf{g}, p-\mu_h\rangle_\Omega = 0 \quad \text{for all } \mu_h \in Q_h.$$

Thus we have that for all $\mathbf{v}_h \in V_h$ and $q_h, \mu_h \in Q_h$,

$$\begin{aligned} \langle \mathbf{g}, \mathbf{u}-\mathbf{u}_h\rangle_{\Omega_h} &= \langle \nabla(\mathbf{v}_\mathbf{g}-\mathbf{v}_h+\mathbf{v}_h), \nabla(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} + \langle \mu_\mathbf{g}, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} \\ &= \langle \nabla(\mathbf{v}_\mathbf{g}-\mathbf{v}_h), \nabla(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} + \langle \operatorname{div} \mathbf{v}_h, p-q_h\rangle \\ &\qquad + \langle \mu_\mathbf{g}, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} \\ &= \langle \nabla(\mathbf{v}_\mathbf{g}-\mathbf{v}_h), \nabla(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} + \langle \operatorname{div}(\mathbf{v}_h-\mathbf{v}_\mathbf{g}), p-q_h\rangle_{\Omega_h} \\ &\qquad + \langle \mu_g, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} \\ &= \langle \nabla(\mathbf{v}_\mathbf{g}-\mathbf{v}_h), \nabla(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} + \langle \operatorname{div}(\mathbf{v}_h-\mathbf{v}_\mathbf{g}), p-q_h\rangle_{\Omega_h} \\ &\qquad + \langle \mu_\mathbf{g}-\mu_h, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} \end{aligned}$$

since $\langle \mu_h, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)\rangle_{\Omega_h} = 0$. We have that for all $\mathbf{v}_h \in X_h$ and $q_h, \mu_h \in Q_h$,

$$\begin{aligned} |\langle \mathbf{g}, \mathbf{u}-\mathbf{u}_h\rangle_{\Omega_h}| &\leqslant \Big(\|\nabla(\mathbf{v}_\mathbf{g}-\mathbf{v}_h)\|_{\Omega_h} + \|\mu_\mathbf{g}-\mu_h\|_{\Omega_h}\Big) \\ &\quad \Big(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{\Omega_h} + \|p-q_h\|_{\Omega_h}\Big). \end{aligned}$$

Thus, taking $\mathbf{v}_h = \mathscr{I}_h\mathbf{v}_\mathbf{g}$ and μ_h the average of $\mu_\mathbf{g}$ on each K, we have by (5.7) that

$$|\langle \mathbf{g}, \mathbf{u}-\mathbf{u}_h\rangle_{\Omega_h}| \leqslant Ch\big(\|\nabla^2 \mathbf{v}_\mathbf{g}\|_\Omega + \|\nabla \mu_\mathbf{g}\|_\Omega\big)\Big(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{\Omega_h} + \|p-p_h\|_{\Omega_h}\Big)$$
$$\leqslant Ch\|\mathbf{g}\|_{\Omega_h}\Big(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{\Omega_h} + \|p-p_h\|_{\Omega_h}\Big).$$

Thus we have by Theorem 5.2 and (5.5) that

$$\|\mathbf{u}-\mathbf{u}_h\|_{\Omega_h} \leqslant Ch^2\big(\|\nabla^2\mathbf{u}\|_\Omega + \|\nabla p\|_\Omega\big),$$

which completes the proof. □

Corollary 5.4. *With the same assumptions in Theorem* 5.2, *we have*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\Omega_h} \leqslant Ch^2.$$

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